Conservation laws and some applications to traffic flows

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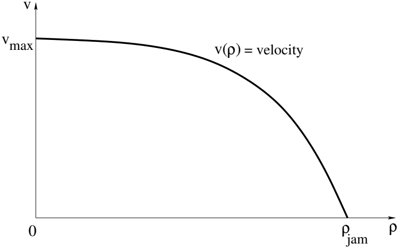
Overview

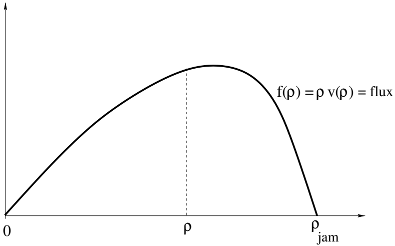
- Review of scalar conservation laws, the Lax formula
- Global optima and Nash equilibria on a network of roads

Cars on a highway

ρ = traffic density = number of cars per unit length

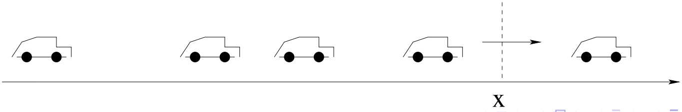
v = velocity of cars (is a decreasing function of the traffic density ρ)



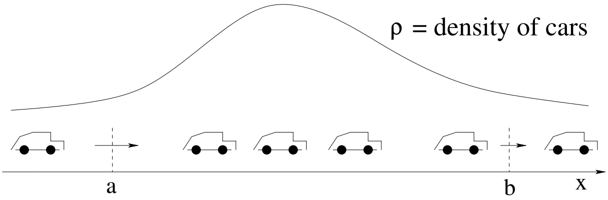


flux: $f(\rho(t,x))$ = [number of cars crossing the point x per unit time]

$$= [\text{density}]\times[\text{velocity}] = \rho(t,x)\cdot v(\rho(t,x))$$



A PDE describing traffic flow

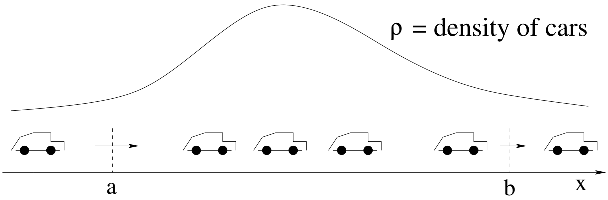


$$\int_a^b \rho(t,x)\,dx \;=\; \text{total number of cars at time } t \text{ within the interval } [a,b]$$

$$\frac{d}{dt}\int_a^b \rho(t,x)\,dx \;=\; [\text{flux of cars entering at } a] - [\text{flux of cars exiting at } b]$$

$$= f(\rho(t,a)) - f(\rho(t,b))$$

A conservation law for the traffic density



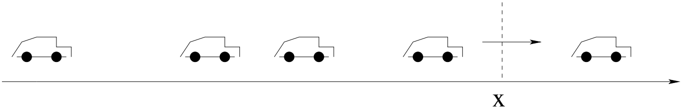
$$\frac{d}{dt}\int_a^b \rho(t,x)\,dx \;=\; \int_a^b \rho_t(t,x)\,dx$$

$$=\; f(\rho(t,a)) - f(\rho(t,b)) \;=\; -\int_a^b \Big[f(\rho(t,x))\Big]_x\,dx$$

$$\int_a^b \Big\{\rho_t + f(\rho)_x\Big\}\,dx \;=\; 0 \qquad\qquad \text{for every} \quad a < b$$

PDE model for traffic flow

Lighthill & Witham (1955), Richards (1956)



$$\begin{cases} \rho_t + f(\rho)_x = 0 & \text{conservation law} \\ \rho(0,x) = \bar{\rho}(x) & \text{initial data} \end{cases}$$

$\rho(t,x)$ = density of cars

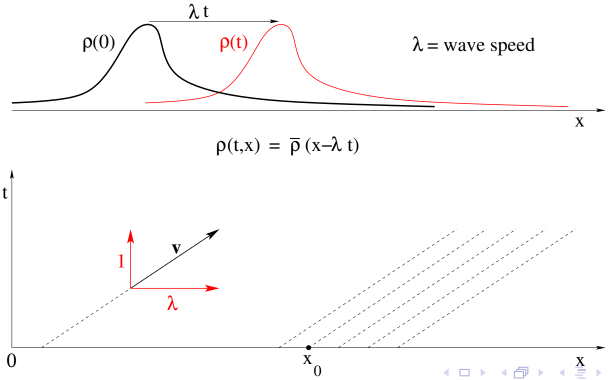
$f(\rho(t,x))$ = flux

Solving a linear wave equation

$$\rho_t + \lambda \rho_x = 0 \qquad \rho(0,x) = \bar{\rho}(x)$$

The directional derivative of $\rho(t,x)$ along the vector $\mathbf{v} = (1, \lambda)$ is zero

$$\rho(t,\, x_0 + \lambda \cdot t) = \bar{\rho}(x_0)$$

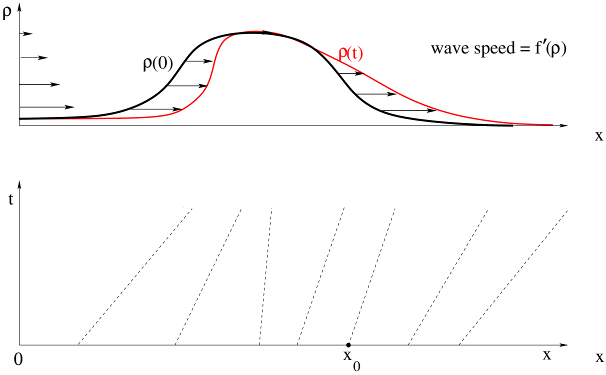


Solving a quasilinear wave equation

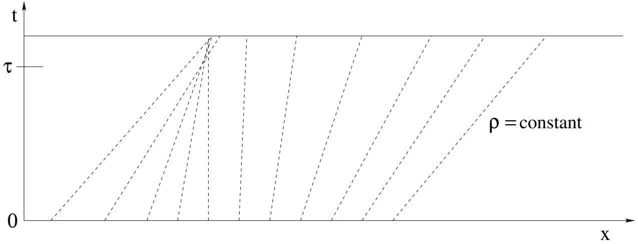
$$\rho_t + f(\rho)_x = 0 \qquad \rho(0,x) = \bar{\rho}(x)$$

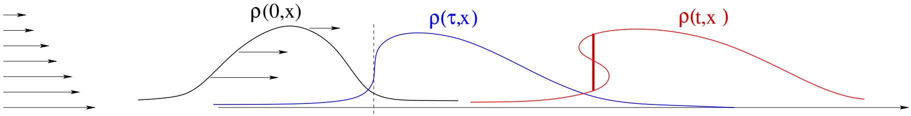
$$\rho_t + f'(\rho)\rho_x = 0 \qquad \text{wave speed} = f'(\rho)$$

$$\rho\Big(t,\, x_0 + f'(\bar{\rho}(x_0))\cdot t\Big) = \bar{\rho}(x_0)$$



Shock formation





Points on the graph of $\rho(t, \cdot)$ move horizontally, with speed $f'(\rho)$

At a finite time τ the tangent becomes vertical and a shock is formed

From conservation laws to Hamilton Jacobi equations

The conservation law

$$\rho_t + f(\rho)_x \ = \ 0\,.$$

Set

$$V(t,x) := \int_{-\infty}^{x} \rho(t,y)dy.$$

The Hamilton Jacobi equation

$$V_t(t,x) + f(V_x(t,x)) = 0\,.$$

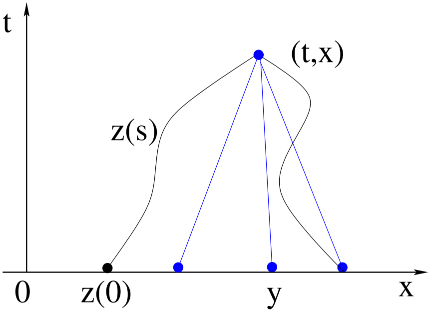
$V(t,x)$ is a value function for an optimization problem.

An optimization problem

$$\rho_t + f(\rho)_x \;=\; 0 \qquad\qquad \rho(0,x) \;=\; \bar\rho(x) \in [0, \rho^{jam}]$$

Legendre transform: $$g(v) \;\doteq\; \inf_{u\in[0,\,\rho^{jam}]} \{vu - f(u)\}$$

Given the initial data $\overline{V}(x) \;\doteq\; \int_{-\infty}^{x} \bar\rho(y)\,dy$ and a terminal point (t,x).

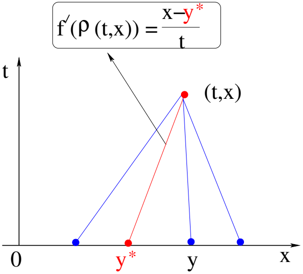


Maximize: $$\overline{V}(z(0)) + \int_0^t g(\dot z(s))\,ds,$$

subject to $z(t) = x$.

The Lax formula

Legendre transform: $$g(v) \doteq \inf_{u \in [0,\, \rho^{jam}]} \{vu - f(u)\}$$



The Lax formula

$$V(t,x) \;=\; \max_{y} \;\; \overline{V}(y) + t\,g\left(\frac{x-y}{t}\right)$$

is the solution of

$$V_t + f(V_x) \;=\; 0 \qquad\qquad \text{for a.e.} \;\; (t,x)$$

Moreover $\rho = V_x$ yields a solution of conservation law

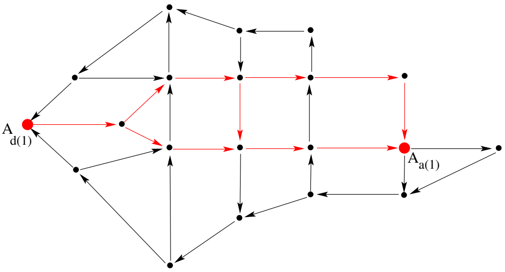
$$\rho_t + f(\rho)_x \;=\; 0\,.$$

Overview

- Review of scalar conservation laws, the Lax formula
- Global optima and Nash equilibria on a network of roads

Traffic flows on a network of roads

- Several groups of drivers, with different departure and arrival locations
- Drivers in the k-th group depart from $A_{d(k)}$ and arrive to $A_{a(k)}$

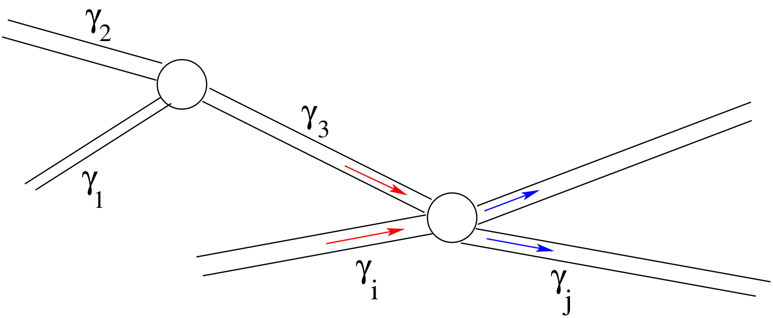


GOAL:

- Describe the evolution of traffic density
- Study optimization problems (for a central planner)
- Study Nash equilibrium problems

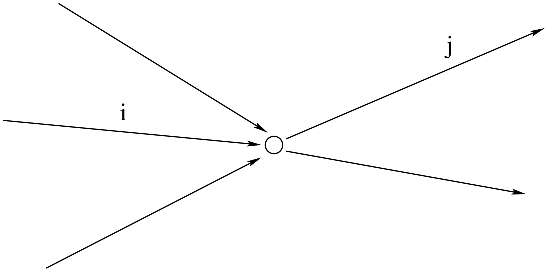
Modeling Traffic Flow on a Network of Roads

- A conservation law on each road γ_k: $\rho_t + f_k(\rho)_x = 0$
- Boundary conditions at intersections. These depend on
  - fraction θ_{ij} of drivers coming from road i who wish to turn into road j
  - priority given to different incoming roads



Intersection Models

Intersection Models (with θ_{ij} constant): M. Garavello & B. Piccoli (2006, 2009).



(A. Bressan & N-, 2014) New intersection models with θ_{ij} variable in time.

Number of vehicles on road i that wish to turn into road j is conserved:

$$(\rho\theta_{ij})_t + (\rho v_i(\rho)\theta_{ij})_x = 0$$

Remark: The basic property of a mathematical model is well posedness.

- M. Garavello - B. Piccoli (2009) prove well posedness for θ_{ij} constant based on Riemann Solvers.
- A. Bressan, F. Yu (2014): Provide several examples on the ill posedness for θ_{ij} variable.
- (A. Bressan & N-, 2015) prove well-posedness with θ_{ij} variable, for a new intersection model including buffers, using a Lax-type representation formula.

Avoiding traffic jams

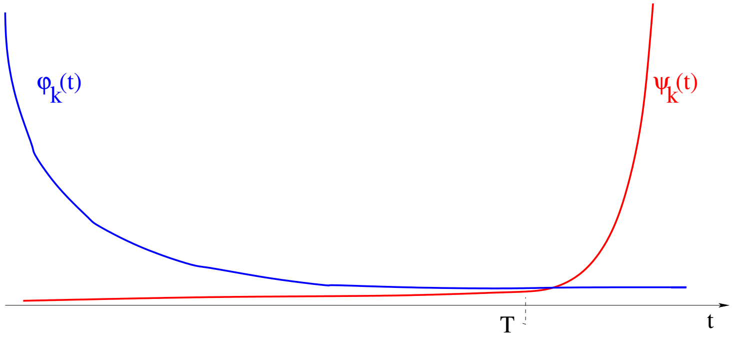
From a practical point of view, the most relevant feature of traffic flow are traffic jams.

It occurs simply because too many drivers choose to be on the same road at the same time.

Look at traffic flow from an engineering point of view, as an optimal decision problem. (deciding the departure time and the route to destination)

An optimal decision problem

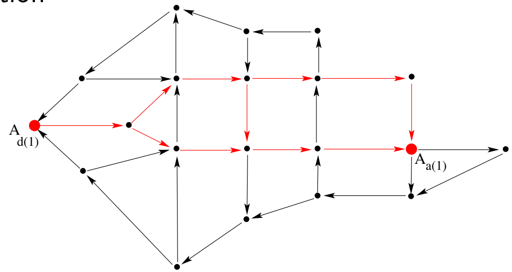
Departure cost: $\varphi_k(t)$ arrival cost: $\psi_k(t)$



$$\varphi_k'(t) < 0 \quad , \quad \psi_k'(t) > 0 \quad \text{and} \quad \lim_{|t|\to\infty} (\varphi_k(t) + \psi_k(t)) = \infty .$$

Optima and equilibria

$\bar{u}_{k,p}(t)$ = rate of departure of drivers of the k-th group, choosing path Γ_p to reach destination



An admissible family $\{\bar{u}_{k,p}\}$ of departure rates is **globally optimal** *if it minimizes the sum of the total costs of all drivers.*

An admissible family $\{u^*_{k,p}\}$ of departure rates is a **Nash equilibrium solution** *if no driver of any group can lower his own total cost by changing departure time or switching to a different path to reach destination.*

Existence results

Thanks to the continuity properties of our new model.

Theorem *(A. Bressan & N-, 2015).*

- On a general network of roads, there exists at least one globally optimal solution $\{\bar{u}_{k,p}\}$.
- If an upper bound on the travel time (for all drivers, under all traffic conditions) is available, then there exists at least one Nash equilibrium solution $\{u^*_{k,p}\}$.

Proof: By finite dimensional approximations + topological methods

No uniqueness, in general

Some open questions

- Find conditions that guarantee that traffic will not get stuck, namely: every driver reaches destination within finite time
- Find necessary conditions for global optima. Study uniqueness. Determine explicit solutions in some simple cases
- Uniqueness and stability of Nash equilibrium.