Variational formulations for the linear viscoelastic problem in the time domain

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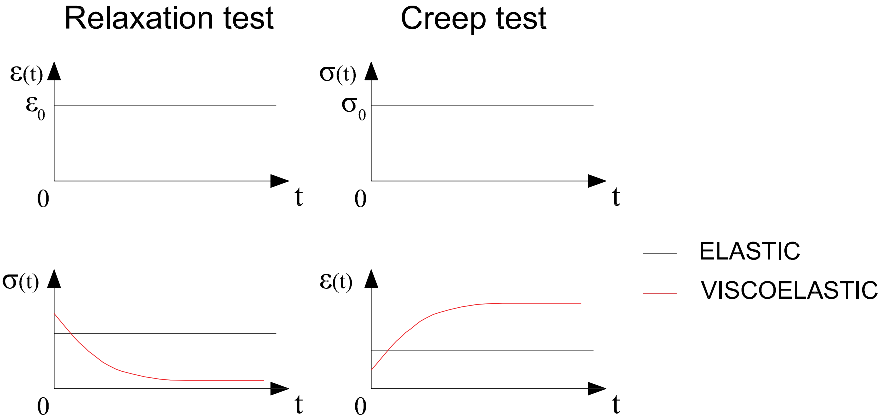
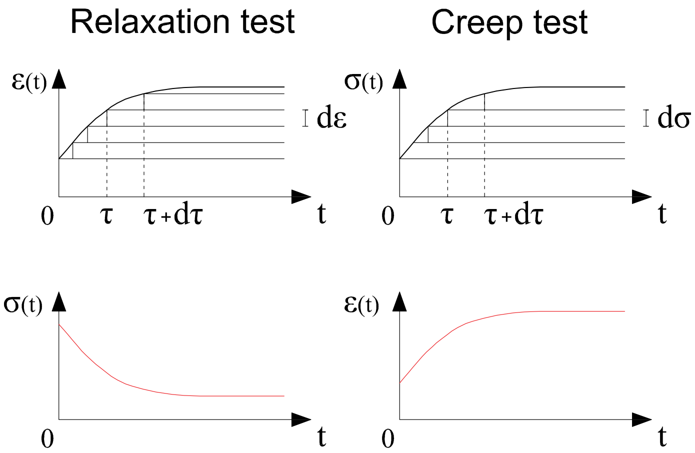
Outline

Viscoelastic vs Elastic materials 1D



$$\sigma(t) = \begin{cases} E\,H(t)\epsilon_0 \\ R(t)\epsilon_0 \end{cases} \qquad \epsilon(t) = \begin{cases} \frac{1}{E}\,H(t)\sigma_0 \\ C(t)\sigma_0 \end{cases}$$

Viscoelasticity: general form



$$\sigma(t) = \int_{0^-}^{t} R(t-\tau)\,\mathrm{d}\epsilon(\tau) \qquad \epsilon(t) = \int_{0^-}^{t} C(t-\tau)\,\mathrm{d}\sigma(\tau)$$

Viscoelasticity 3D

$$\sigma_{ij}(x_r, t) = \int_{0^-}^{t} R_{ijhk}(x_r, t-\tau)\,\mathrm{d}\epsilon_{ij}(x_r, \tau)$$

- Symmetry properties:

$$R_{ijhk}(x_r, t) = R_{jihk}(x_r, t) = R_{ijkh}(x_r, t) = R_{hkij}(x_r, t)$$

- Behaviour at 0 and ∞:

$$R^0_{ijhk}(x_r)\,\gamma_{ij}\,\gamma_{hk} > 0 \qquad R^\infty_{ijhk}(x_r)\,\gamma_{ij}\,\gamma_{hk} > 0$$

$$R^0_{ijhk}(x_r) := \lim_{t\to 0} R_{ijhk}(x_r, t) \qquad R^\infty_{ijhk}(x_r) := \lim_{t\to +\infty} R_{ijhk}(x_r, t)$$

Viscoelasticity 3D

$$\sigma_{ij}(x_r, t) = \int_{0^-}^{t} R_{ijhk}\left(x_r, t-\tau\right)\, \mathrm{d}\epsilon_{hk}(x_r, \tau)$$

$$\epsilon_{ij}(x_r, t) = \int_{0^-}^{t} C_{ijhk}\left(x_r, t-\tau\right)\, \mathrm{d}\sigma_{hk}(x_r, \tau)$$

Bilinear form: $\langle \sigma_{ij}(x_r, t), \epsilon_{ij}(x_r, t) \rangle_c = \int_V \int_{0^-}^{t} \sigma_{ij}(x_r, t-\tau)\, \mathrm{d}\epsilon_{ij}(x_r, \tau)\, \mathrm{d}\Omega$

$\Rightarrow$ Symmetry but NO definiteness $\Rightarrow$???

Governing equations

- Equilibrium equations: $\mathscr{E}(.) = \mathrm{Div}(.)$

$$\begin{cases} \mathscr{E}(\sigma_{ij}(x_r,t)) + b_i(x_r,t) = 0 \quad \text{in } \Omega \times [0,2T] \\ \\ \sigma_{ij}(x_r,t)\, n_j(x_r) = p_i(x_r,t) \quad \text{on } \Gamma_p \times [0,2T] \end{cases}$$

- Compatibility equations: $\mathscr{C}(.) = \frac{1}{2}(\nabla(.) + \nabla^T(.))$

$$\begin{cases} \epsilon_{ij}(x_r,t) = \mathscr{C}(u_i(x_r,t)) \quad \text{in } \Omega \times [0,2T] \\ \\ u_i(x_r,t) = u_i^0(x_r,t) \quad \text{on } \Gamma_u \times [0,2T] \end{cases}$$

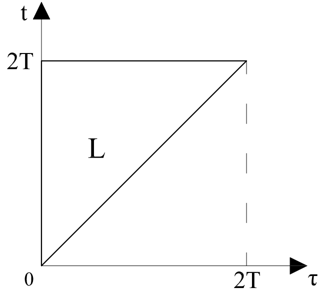
- Constitutive law: $\mathscr{L}\epsilon_{ij}(x_r,\tau) = \int_{0^-}^{t} R_{ijhk}(x_r, t-\tau)\, \mathrm{d}\epsilon_{hk}(x_r,\tau)$

$$\sigma_{ij}(x_r,t) = \mathscr{L}(\epsilon_{ij}(x_r,t)) \quad \text{in } \Omega \times [0,2T]$$

Symmetry of the constitutive law operator

Bilinear form (at a fixed point $x_r \in \Omega$):

$$\langle \sigma'_{ij}, \epsilon''_{ij} \rangle_c := \sigma'_{ij}(2T) * \epsilon''_{ij}(2T) := \int_{0^-}^{2T} \sigma'_{ij}(2T - t)\, \mathrm{d}\epsilon''_{ij}(t)$$



$$\sigma_{ij}(t) = \mathscr{L}\,\epsilon_{ij}(t) = \int_{0^-}^{t} R_{ijhk}(t-\tau)\, \mathrm{d}\epsilon_{hk}(\tau)$$

$$t \in [0, 2T]$$

[see, e.g., Gurtin (1963) and Tonti (1984)]

Decomposition of the time domain

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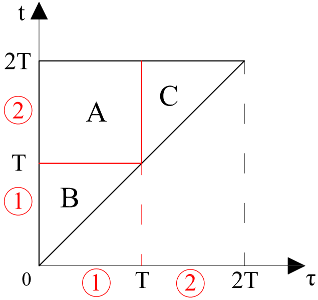

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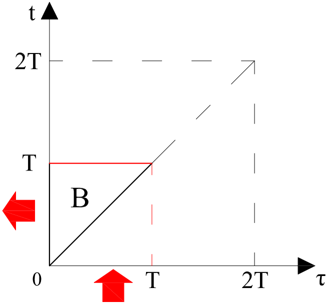

$$\sigma_{1_{ij}}(t) = B\epsilon_{1_{ij}}(t) = \int_{0^-}^{t} R_{ijhk}(t - \tau)\, \mathrm{d}\epsilon_{1_{hk}}(\tau)$$

$$t \in [0, T]$$

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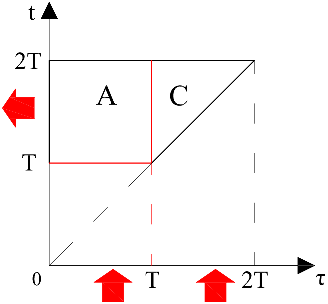

$$\sigma_{2_{ij}}(t) = A\epsilon_{1_{ij}}(t) + C\epsilon_{2_{ij}}(t) \quad t \in [T, 2T]$$

$$A\epsilon_{1_{ij}}(t) = \int_{0^-}^{T} R_{ijhk}(t-\tau)\, \mathrm{d}\epsilon_{1_{hk}}(\tau)$$

$$C\epsilon_{2_{ij}}(t) = \int_{T}^{t} R_{ijhk}(t-\tau)\, \mathrm{d}\epsilon_{2_{hk}}(\tau)$$

Decomposition of the constitutive law operator

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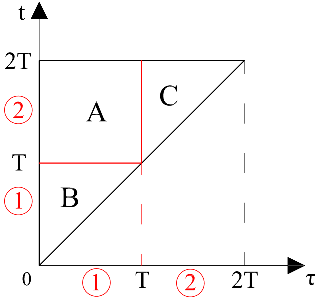

A is symmetric

C is the adjoint operator of B ($\Rightarrow \widetilde{B}$)

$$\begin{bmatrix} \sigma_{2_{ij}}(t) \\ \sigma_{1_{ij}}(t) \end{bmatrix} = \begin{bmatrix} A & \widetilde{B} \\ B & 0 \end{bmatrix} \begin{bmatrix} \epsilon_{1_{ij}}(t) \\ \epsilon_{2_{ij}}(t) \end{bmatrix}$$

Discussion about the free energy functional

$$\frac{1}{2}\langle A\,\epsilon'_{1_{ij}},\epsilon'_{1_{ij}}\rangle_c = \frac{1}{2}\int_{0^-}^{T}\int_{0^-}^{T} R_{ijhk}(2T-t-\tau)\,\mathrm{d}\epsilon'_{1_{hk}}(\tau)\,\mathrm{d}\epsilon'_{1_{ij}}(t)$$

that is the Helmholtz free energy $\mathcal{Q}\,(x_r,\,T)$ in isothermal conditions

[see Staverman and Schwarzl (1952), Bland (1960), Hunter (1961), and Breuer and Onat (1964)]

$\Rightarrow \mathcal{Q}\,(x_r,\,T) \geqslant 0 \Rightarrow$ *A is positive semi-definite*

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Integrated dissipation inequality [Coleman (1964)]:

$$\int_{t_0}^{t_1} \sigma_{ij}(\tau)\,\dot{\epsilon}_{ij}(\tau)\,\mathrm{d}\tau \geqslant \mathcal{Q}(t_1) - \mathcal{Q}(t_0)$$

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If $R_{ijhk}(t)$ is completely monotonic, i.e.

$$(-1)^n R^{(n)}_{ijhk}(t)\,\gamma_{ij}\,\gamma_{hk} \geqslant 0 \quad n = 0, 1, 2, \ldots$$

then $\mathcal{Q}$ is a free energy [Del Piero and Deseri (1996)]

Positive definiteness of the operator A

[Mandel (1960), Huet (2011), Del Piero and Deseri (1997)]:

$$R_{ijhk}(t) \text{ completely monotonic} \quad \Leftrightarrow \quad R_{ijhk}(t) = \int_0^\infty \phi_{ijhk}(\alpha)\, \mathrm{e}^{-\alpha\, t} \mathrm{d}\alpha$$

$$\Rightarrow \quad \mathcal{Q} = \frac{1}{2} \int_0^\infty \phi_{ijhk}(\alpha)\, g_{hk}(\alpha)\, g_{ij}(\alpha)\, \mathrm{d}\alpha$$

where

$$g_{ij}(\alpha) := \int_{0^-}^{T} \mathrm{e}^{-\alpha(T-t)} \mathrm{d}\epsilon'_{1_{ij}}(t)$$

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$$g_{ij}(\alpha) := \int_{0^-}^{T} \mathrm{e}^{-\alpha(T-t)} \mathrm{d}\epsilon'_{1_{ij}}(t)$$

Due to the positive definiteness of $\phi_{ijhk}(\alpha)$,

$$\phi_{ijhk}(\alpha)\, g_{hk}(\alpha)\, g_{ij}(\alpha) > 0$$

$\Rightarrow$ A is positive definite

Definiteness of the new constitutive law operator

The constitutive law

$$\begin{bmatrix} \sigma_{2_{ij}}(t) \\ \sigma_{1_{ij}}(t) \end{bmatrix} = \begin{bmatrix} A & \widetilde{B} \\ B & 0 \end{bmatrix} \begin{bmatrix} \epsilon_{1_{ij}}(t) \\ \epsilon_{2_{ij}}(t) \end{bmatrix} \quad \Leftrightarrow \quad \boldsymbol{\sigma} = \mathbf{L}\,\boldsymbol{\epsilon}$$

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can be rephrased, through a partial Legendre transform [see, e.g., Milton (1990), Cherkaev and Gibiansky (1994)], as follows:

$$\begin{bmatrix} -\sigma_{1_{ij}}(t) \\ \epsilon_{1_{ij}}(t) \end{bmatrix} = \begin{bmatrix} \widetilde{B}A^{-1}B & -\widetilde{B}A^{-1} \\ -A^{-1}B & A^{-1} \end{bmatrix} \begin{bmatrix} \epsilon_{2_{ij}}(t) \\ \sigma_{2_{ij}}(t) \end{bmatrix} \quad \Leftrightarrow \quad \theta_1 = \mathbf{S}\,\theta_2$$

S is positive semi-definite

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Remark. The causality principle is not violated!!!

[see, e.g., Fabrizio et al. (2010), Arch. Rational Mech. Anal. **198**, 189-232]

The decomposition of the inverse constitutive law

$$\epsilon_{1_{ij}}(t) = \mathcal{B}\sigma_{1_{ij}}(t) = \int_{0^-}^{t} C_{ijhk}(t-\tau)\,\mathrm{d}\sigma_{1_{hk}}(\tau) \quad \text{for } t \in [0, T]$$

$$\begin{aligned}\epsilon_{2_{ij}}(t) = \mathcal{A}\sigma_{1_{ij}}(t) + \widetilde{\mathcal{B}}\sigma_{2_{ij}}(t) &= \int_{0^-}^{T} C_{ijhk}(t-\tau)\,\mathrm{d}\sigma_{1_{hk}}(\tau) \\ &\quad + \int_{T}^{t} C_{ijhk}(t-\tau)\,\mathrm{d}\sigma_{2_{hk}}(\tau) \quad \text{for } t \in [T, 2T]\end{aligned}$$

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$$\epsilon_{2_{ij}}(t) = \mathcal{A}\sigma_{1_{ij}}(t) + \widetilde{\mathcal{B}}\sigma_{2_{ij}}(t) = \int_{0^-}^{T} C_{ijhk}(t-\tau)\,\mathrm{d}\sigma_{1_{hk}}(\tau) + \int_{T}^{t} C_{ijhk}(t-\tau)\,\mathrm{d}\sigma_{2_{hk}}(\tau) \quad \text{for } t \in [T, 2T]$$

$$\begin{bmatrix} \epsilon_{1_{ij}}(t) \\ \epsilon_{2_{ij}}(t) \end{bmatrix} = \begin{bmatrix} 0 & \widetilde{\mathcal{B}} \\ \mathcal{B} & \mathcal{A} \end{bmatrix} \begin{bmatrix} \sigma_{2_{ij}}(t) \\ \sigma_{1_{ij}}(t) \end{bmatrix} \quad \Leftrightarrow \quad \boldsymbol{\epsilon} = \mathbf{L}^{-1}\boldsymbol{\sigma}$$

$$\mathcal{B} = B^{-1}$$

$$\mathcal{A} = -B^{-1} A\, \widetilde{B}^{-1} \quad \Rightarrow \quad \mathcal{A} \text{ is negative definite}$$

Six-fields formulation

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -\mathscr{E} \\ 0 & 0 & 0 & 0 & -\mathscr{E} & 0 \\ 0 & 0 & A & \widetilde{B} & 0 & -I \\ 0 & 0 & B & 0 & -I & 0 \\ 0 & \mathscr{C} & 0 & -I & 0 & 0 \\ \mathscr{C} & 0 & -I & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{1_i} \\ u_{2_i} \\ \epsilon_{1_{ij}} \\ \epsilon_{2_{ij}} \\ \sigma_{1_{ij}} \\ \sigma_{2_{ij}} \end{bmatrix} = \begin{bmatrix} b_{2_i} \\ b_{1_i} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} \text{in } \Omega \times [T, 2T] \\ \text{in } \Omega \times [0, T] \\ \text{in } \Omega \times [T, 2T] \\ \text{in } \Omega \times [0, T] \\ \text{in } \Omega \times [T, 2T] \\ \text{in } \Omega \times [0, T] \end{matrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & n_j \\ 0 & 0 & 0 & 0 & n_j & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -n_j & 0 & 0 & 0 & 0 \\ -n_j & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{1_i} \\ u_{2_i} \\ \epsilon_{1_{ij}} \\ \epsilon_{2_{ij}} \\ \sigma_{1_{ij}} \\ \sigma_{2_{ij}} \end{bmatrix} = \begin{bmatrix} p_{2_i} \\ p_{1_i} \\ 0 \\ 0 \\ -n_j u^0_{2_i} \\ -n_j u^0_{1_i} \end{bmatrix} \quad \begin{matrix} \text{on } \Gamma_p \times [T, 2T] \\ \text{on } \Gamma_p \times [0, T] \\ \\ \\ \text{on } \Gamma_u \times [T, 2T] \\ \text{on } \Gamma_u \times [0, T] \end{matrix}$$

$$\mathbf{M}_I \, \mathbf{z}_I = \mathbf{b}_I \quad \text{in } \Omega \times [0, 2T]$$

$$\mathbf{T}_I \, \mathbf{z}_I = \mathbf{g}_I \quad \text{on } \Gamma \times [0, 2T]$$

Four-fields formulation

$$\begin{bmatrix} 0 & 0 & 0 & \mathscr{E} \\ 0 & 0 & \mathscr{E} & 0 \\ 0 & -\mathscr{C} & \mathcal{A} & \widetilde{\mathcal{B}} \\ -\mathscr{C} & 0 & \mathcal{B} & 0 \end{bmatrix} \begin{bmatrix} u_{1_i} \\ u_{2_i} \\ \sigma_{1_{ij}} \\ \sigma_{2_{ij}} \end{bmatrix} = \begin{bmatrix} -b_{2_i} \\ -b_{1_i} \\ 0 \\ 0 \end{bmatrix} \quad \begin{matrix} \text{in } \Omega \times [T, 2T] \\ \text{in } \Omega \times [0, T] \\ \text{in } \Omega \times [T, 2T] \\ \text{in } \Omega \times [0, T] \end{matrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & -n_j \\ 0 & 0 & -n_j & 0 \\ 0 & n_j & 0 & 0 \\ n_j & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{1_i} \\ u_{2_i} \\ \sigma_{1_{ij}} \\ \sigma_{2_{ij}} \end{bmatrix} = \begin{bmatrix} -p_{2_i} \\ -p_{1_i} \\ n_j u^0_{2_i} \\ n_j u^0_{1_i} \end{bmatrix} \quad \begin{matrix} \text{on } \Gamma_p \times [T, 2T] \\ \text{on } \Gamma_p \times [0, T] \\ \text{on } \Gamma_u \times [T, 2T] \\ \text{on } \Gamma_u \times [0, T] \end{matrix}$$

$$\mathbf{M}_{II}\, \mathbf{z}_{II} = \mathbf{b}_{II} \quad \text{in } \Omega \times [0, 2T]$$

$$\mathbf{T}_{II}\, \mathbf{z}_{II} = \mathbf{g}_{II} \quad \text{on } \Gamma \times [0, 2T]$$

Five-fields formulation

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \mathscr{E} \\ 0 & 0 & 0 & \mathscr{E} & 0 \\ 0 & 0 & \widetilde{B}A^{-1}B & -\widetilde{B}\,A^{-1} & I \\ 0 & -\mathscr{C} & -A^{-1}B & A^{-1} & 0 \\ -\mathscr{C} & 0 & I & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{2_i} \\ u_{1_i} \\ \epsilon_{2_{ij}} \\ \sigma_{2_{ij}} \\ \sigma_{1_{ij}} \end{bmatrix} = \begin{bmatrix} -b_{1_i} \\ -b_{2_i} \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{matrix} \text{in } \Omega \times [0, T] \\ \text{in } \Omega \times [T, 2T] \\ \text{in } \Omega \times [0, T] \\ \text{in } \Omega \times [0, T] \\ \text{in } \Omega \times [T, 2T] \end{matrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & -n_j \\ 0 & 0 & 0 & -n_j & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & n_j & 0 & 0 & 0 \\ n_j & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{2_i} \\ u_{1_i} \\ \epsilon_{2_{ij}} \\ \sigma_{2_{ij}} \\ \sigma_{1_{ij}} \end{bmatrix} = \begin{bmatrix} -p_{1_i} \\ -p_{2_i} \\ 0 \\ n_j u^0_{1_i} \\ n_j u^0_{2_i} \end{bmatrix} \begin{matrix} \text{on } \Gamma_p \times [0, T] \\ \text{on } \Gamma_p \times [T, 2T] \\ \\ \text{on } \Gamma_u \times [0, T] \\ \text{on } \Gamma_u \times [T, 2T] \end{matrix}$$

$$\mathbf{M}_{III}\, \mathbf{z}_{III} = \mathbf{b}_{III} \quad \text{in } \Omega \times [0, 2T]$$

$$\mathbf{T}_{III}\, \mathbf{z}_{III} = \mathbf{g}_{III} \quad \text{on } \Gamma \times [0, 2T]$$

Variational formulations

$$\begin{array}{ll} \mathbf{M}_i\,\mathbf{z}_i = \mathbf{b}_i & \text{in } \Omega \times [0, 2T] \\ \mathbf{T}_i\,\mathbf{z}_i = \mathbf{g}_i & \text{on } \Gamma \times [0, 2T] \end{array} \quad \Rightarrow \quad \mathbf{N}_i\,\mathbf{z}_i = \mathbf{f}_i \quad i = I,\ II,\ III$$

$$\langle\langle \mathbf{N}_i\,\mathbf{z}_i', \mathbf{z}_i'' \rangle\rangle_c = \int_\Omega \mathbf{M}_i\,\mathbf{z}_i'(2T) * \mathbf{z}_i''(2T)\,\mathrm{d}\Omega + \int_\Gamma \mathbf{T}_i\,\mathbf{z}_i'(2T) * \mathbf{z}_i''(2T)\,\mathrm{d}\Gamma$$

Variational formulations

$$\begin{array}{ll}\mathbf{M}_i\,\mathbf{z}_i = \mathbf{b}_i & \text{in } \Omega \times [0, 2T] \\ \mathbf{T}_i\,\mathbf{z}_i = \mathbf{g}_i & \text{on } \Gamma \times [0, 2T]\end{array} \quad \Rightarrow \quad \mathbf{N}_i\,\mathbf{z}_i = \mathbf{f}_i \quad i = I,\ II,\ III$$

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$$\mathbf{N}_i\,\mathbf{z}_i = \mathbf{f}_i \quad \Leftrightarrow \quad \mathcal{F}_i(\mathbf{z}_i) = \operatorname*{stat}_{\mathbf{z}_i'} \mathcal{F}_i(\mathbf{z}_i')$$

$$\begin{aligned}\mathcal{F}_i(\mathbf{z}_i') =\ & \frac{1}{2}\langle\langle \mathbf{N}_i\,\mathbf{z}_i', \mathbf{z}_i'\rangle\rangle_c - \langle\langle \mathbf{f}_i, \mathbf{z}_i'\rangle\rangle_c \\ =\ & \frac{1}{2}\int_\Omega \mathbf{M}_i\,\mathbf{z}_i'(2T) * \mathbf{z}_i'(2T)\,\mathrm{d}\Omega + \frac{1}{2}\int_\Gamma \mathbf{T}_i\,\mathbf{z}_i'(2T) * \mathbf{z}_i'(2T)\,\mathrm{d}\Gamma \\ & - \int_\Omega \mathbf{b}_i(2T) * \mathbf{z}_i'(2T)\,\mathrm{d}\Omega - \int_\Gamma \mathbf{g}_i(2T) * \mathbf{z}_i'(2T)\,\mathrm{d}\Gamma\end{aligned}$$

Functional of the Total Potential Energy type

By imposing the strain-displacement relations into the six-fields variational formulation:

$$\begin{aligned}\mathrm{TPE}(u'_{1_i}, u'_{2_i}) = &\frac{1}{2}\int_\Omega \Big(A\,\mathscr{C}\, u'_{1_i}(2T) * \mathscr{C}\, u'_{1_i}(2T) + 2\widetilde{B}\,\mathscr{C}\, u'_{1_i}(2T) * \mathscr{C}\, u'_{2_i}(2T) \\ &- \int_\Omega \Big(b_{2_i}(2T) * u'_{1_i}(2T) + b_{1_i}(2T) * u'_{2_i}(2T) \Big)\, \mathrm{d}\Omega \\ &- \int_{\Gamma_p} \Big(p_{2_i}(2T) * u'_{1_i}(2T) + p_{1_i}(2T) * u'_{2_i}(2T) \Big)\, \mathrm{d}\Gamma\end{aligned}$$

$$\mathrm{TPE}(u_{1_i}, u_{2_i}) = \min_{u'_{1_i}} \operatorname*{stat}_{u'_{2_i}} \mathrm{TPE}(u'_{1_i}, u'_{2_i})$$

u'_{1_i} and u'_{2_i} being compatible displacement fields

Functional of the Total Complementary Energy type

By imposing the equilibrium equations into the four-fields variational formulation:

$$\mathrm{TCE}(\sigma'_{1_{ij}}, \sigma'_{2_{ij}}) = \frac{1}{2}\int_\Omega \left(\mathcal{A}\,\sigma'_{1_{ij}}(2T) * \sigma'_{1_{ij}}(2T) + 2\,\widetilde{\mathcal{B}}\,\sigma'_{1_{ij}}(2T) * \sigma'_{2_{ij}}(2T) \right) \mathrm{d}\Omega$$
$$- \int_{\Gamma_u} \left(n_j\, u^0_{2_i}(2T) * \sigma'_{1_{ij}}(2T) + n_j\, u^0_{1_i}(2T) * \sigma'_{2_{ij}}(2T) \right) \mathrm{d}\Gamma$$

$$\mathrm{TCE}(\sigma_{1_{ij}}, \sigma_{2_{ij}}) = \max_{\sigma'_{1_{ij}}} \operatorname*{stat}_{\sigma'_{2_{ij}}} \mathrm{TCE}(\sigma'_{1_{ij}}, \sigma'_{2_{ij}})$$

$\sigma'_{1_{ij}}$ and $\sigma'_{2_{ij}}$ being equilibrated stress fields

New functional

$$
\begin{aligned}
\mathrm{F}(\sigma'_{2_{ij}}, u'_{2_i}) = \frac{1}{2} \int_\Omega \Big(& A^{-1} \sigma'_{2_{ij}}(2T) * \sigma'_{2_{ij}}(2T) - \widetilde{B}\, A^{-1}\, \sigma'_{2_{ij}}(2T) * \mathscr{C} u'_{2_i}(2T) \\
& + \widetilde{B}\, A^{-1}\, B\, \mathscr{C}\, u'_{2_i}(2T) * \mathscr{C}\, u'_{2_i}(2T) \\
& - A^{-1}\, B\, \mathscr{C}\, u'_{2_i}(2T) * \sigma'_{2_{ij}}(2T) \Big) \,\mathrm{d}\Omega \\
+ & \int_\Omega b_{1_i}(2T) * u'_{2_i}(2T) \,\mathrm{d}\Omega \\
+ & \int_{\Gamma_p} p_{1_i}(2T) * u'_{2_i}(2T) \,\mathrm{d}\Gamma - \int_{\Gamma_u} n_j\, u^0_{1_i}(2T) * \sigma'_{2_{ij}}(2T) \,\mathrm{d}\Gamma
\end{aligned}
$$

$$
\mathrm{F}(\sigma_{2_{ij}}, u_{2_i}) = \min_{u'_{2_i}, \sigma'_{2_{ij}}} \mathrm{F}(\sigma'_{2_{ij}}, u'_{2_i})
$$

u'_{2_i} and $\sigma'_{2_{ij}}$ being, respectively, a compatible displacement field and an equilibrated stress field

The problem on the RVE

$$\begin{cases} \sigma_{ij/j} = 0 \quad \text{in } \Omega \times [0, 2T] \\ \epsilon_{ij} = \frac{1}{2}\left(u_{i/j} + u_{j/i}\right) \quad \text{in } \Omega \times [0, 2T] \\ u_i = \overline{\epsilon}_{ij} x_j \quad \text{on } \Gamma \times [0, 2T] \\ \sigma_{ij} = \mathcal{L}\epsilon_{ij} \quad \text{in } \Omega \times [0, 2T] \end{cases}$$

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$$\overline{\sigma}_{ij}(t) = \int_{0^-}^{t} R^h_{ijhk}(t-\tau)\, \mathrm{d}\overline{\epsilon}_{hk}(\tau) \quad \text{for } t \in [0, 2T]$$

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$$\begin{bmatrix} \overline{\sigma}_{2_{ij}}(t) \\ \overline{\sigma}_{1_{ij}}(t) \end{bmatrix} = \begin{bmatrix} A^h & \widetilde{B}^h \\ B^h & 0 \end{bmatrix} \begin{bmatrix} \overline{\epsilon}_{1_{ij}}(t) \\ \overline{\epsilon}_{2_{ij}}(t) \end{bmatrix} \quad \Leftrightarrow \quad \overline{\boldsymbol{\sigma}} = \mathbf{L}^h \, \overline{\boldsymbol{\epsilon}}$$

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$$\begin{bmatrix} -\overline{\sigma}_{1_{ij}}(t) \\ \overline{\epsilon}_{1_{ij}}(t) \end{bmatrix} = \begin{bmatrix} \widetilde{B}^h (A^h)^{-1} B^h & -\widetilde{B}^h (A^h)^{-1} \\ -(A^h)^{-1} B^h & (A^h)^{-1} \end{bmatrix} \begin{bmatrix} \overline{\epsilon}_{2_{ij}}(t) \\ \overline{\sigma}_{2_{ij}}(t) \end{bmatrix} \; \Leftrightarrow \; \overline{\theta}_1 = \mathbf{S}^h\, \overline{\theta}_2$$

Bounds

$$\mathrm{F}(\sigma'_{2_{ij}}, \epsilon'_{2_{ij}}) = \frac{1}{2}\int_\Omega \Big(\epsilon'_{2_{ij}}(2T) * \widetilde{B} A^{-1} B\, \epsilon'_{2_{ij}}(2T) - 2\sigma'_{2_{ij}}(2T) * A^{-1} B \epsilon'_{2_{ij}}(2T) + \sigma'_{2_{ij}}(2T) * A^{-1} \sigma'_{2_{ij}}(2T) \Big) \mathrm{d}\Omega - V\, \overline{\sigma'}_{2_{ij}}(2T) * \overline{\epsilon}_{1_{ij}}(2T)$$

$$\mathrm{F}(\sigma_{2_{ij}}, \epsilon_{2_{ij}}) \leqslant \mathrm{F}(\overline{\sigma}_{2_{ij}}, \overline{\epsilon}_{2_{ij}})$$

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$$+ \sigma'_{2_{ij}}(2T) * A^{-1}\sigma'_{2_{ij}}(2T)\Big)\,\mathrm{d}\Omega - V\,\overline{\sigma'}_{2_{ij}}(2T) * \overline{\epsilon}_{1_{ij}}(2T)$$

$$\mathrm{F}(\sigma_{2_{ij}}, \epsilon_{2_{ij}}) \leqslant \mathrm{F}(\overline{\sigma}_{2_{ij}}, \overline{\epsilon}_{2_{ij}})$$

In the solution

$$\frac{1}{2}\int_\Omega \Big(\epsilon_{2_{ij}}(2T) * \widetilde{B}\, A^{-1}\, B\, \epsilon_{2_{ij}}(2T) - 2\,\sigma_{2_{ij}}(2T) * A^{-1}\, B\, \epsilon_{2_{ij}}(2T)$$
$$+ \sigma_{2_{ij}}(2T) * A^{-1}\sigma_{2_{ij}}(2T)\Big)\,\mathrm{d}\Omega = \frac{1}{2}\int_\Omega \boldsymbol{\theta_2}(2T) * \mathbf{S}\,\boldsymbol{\theta_2}(2T)\,\mathrm{d}\Omega$$

Since $\mathbf{S}\,\boldsymbol{\theta_2} = \boldsymbol{\theta_1}$ and by virtue of Hill's principle:

$$\frac{1}{2}\int_\Omega \boldsymbol{\theta_2}(2T) * \mathbf{S}\,\boldsymbol{\theta_2}(2T)\,\mathrm{d}\Omega = \frac{V}{2}\,\overline{\boldsymbol{\theta}}_2(2T) * \overline{\boldsymbol{\theta}}_\mathbf{1}(2T)$$

Bounds

Recalling that $\overline{\theta}_1 = \mathbf{S}^h \, \overline{\theta}_2$:

$$\mathrm{F}(\sigma_{2_{ij}}, \epsilon_{2_{ij}}) = \frac{V}{2}\Bigg(\overline{\epsilon}_{2_{ij}}(2T) * \widetilde{B}^h \, (A^h)^{-1} \, B^h \, \overline{\epsilon}_{2_{ij}}(2T) - 2 \, \overline{\sigma}_{2_{ij}}(2T) * (A^h)^{-1} \, B^h \, \overline{\epsilon}_{2_{ij}}(2T) + \overline{\sigma}_{2_{ij}}(2T) * (A^h)^{-1} \overline{\sigma}_{2_{ij}}(2T) \Bigg) - V \, \overline{\sigma}_{2_{ij}}(2T) * \overline{\epsilon}_{1_{ij}}(2T)$$

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In correspondence to the average fields:

$$\mathrm{F}(\overline{\sigma}_{2_{ij}}, \overline{\epsilon}_{2_{ij}}) = \frac{V}{2}\Bigg(\overline{\epsilon}_{2_{ij}}(2T) * \overline{\widetilde{B} A^{-1} B}\, \overline{\epsilon}_{2_{ij}}(2T) - 2\,\overline{\sigma}_{2_{ij}}(2T) * \overline{A^{-1} B}\, \overline{\epsilon}_{2_{ij}}(2T) + \overline{\sigma}_{2_{ij}}(2T) * \overline{A^{-1}}\, \overline{\sigma}_{2_{ij}}(2T) \Bigg) - V\,\overline{\sigma}_{2_{ij}}(2T) * \overline{\epsilon}_{1_{ij}}(2T)$$

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Bounds

$$\overline{\epsilon}_{2_{ij}}(2T) * \widetilde{B}^h (A^h)^{-1} B^h \, \overline{\epsilon}_{2_{ij}}(2T) - 2\, \overline{\epsilon}_{2_{ij}}(2T) * \widetilde{B}^h (A^h)^{-1} \overline{\sigma}_{2_{ij}}(2T)$$
$$+ \, \overline{\sigma}_{2_{ij}}(2T) * (A^h)^{-1} \overline{\sigma}_{2_{ij}}(2T) \leqslant \overline{\epsilon}_{2_{ij}}(2T) * \overline{\widetilde{B} A^{-1} B} \, \overline{\epsilon}_{2_{ij}}(2T)$$
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Remark 1. Analogy with the bounds obtained by Cherkaev and Gibiansky (1994) in the frequency domain

Bounds

$$
\begin{aligned}
&\overline{\epsilon}_{2_{ij}}(2T) * \widetilde{B}^h (A^h)^{-1} B^h \, \overline{\epsilon}_{2_{ij}}(2T) - 2\, \overline{\epsilon}_{2_{ij}}(2T) * \widetilde{B}^h (A^h)^{-1} \overline{\sigma}_{2_{ij}}(2T) \\
&\quad + \overline{\sigma}_{2_{ij}}(2T) * (A^h)^{-1} \overline{\sigma}_{2_{ij}}(2T) \leqslant \overline{\epsilon}_{2_{ij}}(2T) * \overline{\widetilde{B} A^{-1} B}\, \overline{\epsilon}_{2_{ij}}(2T) \\
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\end{aligned}
$$

Remark 1. Analogy with the bounds obtained by Cherkaev and Gibiansky (1994) in the frequency domain

Remark 2. $\overline{\sigma}_{2_{ij}}(t) = A^h \, \overline{\epsilon}_{1_{ij}}(t) + B^h \, \overline{\epsilon}_{2_{ij}}(t)$

$$
\overline{\sigma}_{2_{ij}} = 0 \Rightarrow \overline{\epsilon}_{2_{ij}}(2T) * \widetilde{B}^h (A^h)^{-1} B^h \, \overline{\epsilon}_{2_{ij}}(2T) \leqslant \overline{\epsilon}_{2_{ij}}(2T) * \overline{\widetilde{B} A^{-1} B}\, \overline{\epsilon}_{2_{ij}}(2T)
$$

Bounds

$$\overline{\epsilon}_{2_{ij}}(2T) * \widetilde{B}^h (A^h)^{-1} B^h \,\overline{\epsilon}_{2_{ij}}(2T) - 2\,\overline{\epsilon}_{2_{ij}}(2T) * \widetilde{B}^h (A^h)^{-1} \overline{\sigma}_{2_{ij}}(2T) + \overline{\sigma}_{2_{ij}}(2T) * (A^h)^{-1} \overline{\sigma}_{2_{ij}}(2T) \leqslant \overline{\epsilon}_{2_{ij}}(2T) * \overline{\widetilde{B} A^{-1} B}\, \overline{\epsilon}_{2_{ij}}(2T) - 2\,\overline{\epsilon}_{2_{ij}}(2T) * \overline{\widetilde{B} A^{-1}}\, \overline{\sigma}_{2_{ij}}(2T) + \overline{\sigma}_{2_{ij}}(2T) * \overline{A^{-1}}\, \overline{\sigma}_{2_{ij}}(2T)$$

Remark 1. Analogy with the bounds obtained by Cherkaev and Gibiansky (1994) in the frequency domain

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$$\overline{\epsilon}_{2_{ij}} = 0 \quad \Rightarrow \quad \overline{\sigma}_{2_{ij}}(2T) * (A^h)^{-1} \,\overline{\sigma}_{2_{ij}}(2T) \leqslant \overline{\sigma}_{2_{ij}}(2T) * \overline{A^{-1}}\, \overline{\sigma}_{2_{ij}}(2T)$$

Conclusions

Convolutive bilinear form

$+$

Decomposition of the time domain into two equal subintervals

$+$

Partial Legendre transform

$=$

Minimum variational formulation

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Convolutive bilinear form

$+$

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$+$

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"In simple relaxation or creep experiments the thermodynamic functions at time t depend on the relaxation or creep function at time 2t." [Staverman and Schwarzl (1952)]

Open issues

- Relation with the analogous bounds in the frequency domain [Cherkaev and Gibiansky (1994)]
- Applications to Elastodynamics, Heat conduction, Instability...

Thank you for your attention