Introduction to Arbitrage-Free Pricing: Fundamental Theorems

Dmitry Kramkov

Carnegie Mellon University

Workshop on Interdisciplinary Mathematics,
Penn State, May 8-10, 2015

Outline

Financial market

Pricing $=$ Replication

Black and Scholes formula

Fundamental theorems

Martingale Representation

Summary

ISDA Market Survey

Notional amounts outstanding at year-end, all surveyed contracts, 1987-present

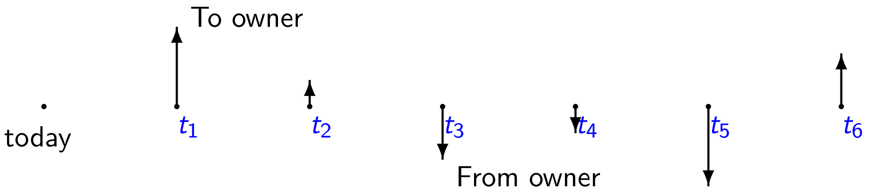
Notional amounts in billions of US dollars

	Year-end outstandings for interest rate swaps	Year-end outstandings for currency swaps	Year-end outstandings for interest rate options	Total IR and currency outstandings	Total credit default swap outstandings	Total equity derivative outstandings
1987	$ 682.80	$ 182.80		$ 865.60		
1988	1,010.20	316.80	327.30	1,654.30		
1989	1,502.60	434.90	537.30	2,474.70		
1990	2,311.54	577.53	561.30	3,450.30		
1991	3,065.10	807.67	577.20	4,449.50		
1992	3,850.81	860.39	634.50	5,345.70		
1993	6,177.35	899.62	1,397.60	8,474.50		
1994	8,815.56	914.85	1,572.80	11,303.20		
1995	12,810.74	1,197.39	3,704.50	17,712.60		
1996	19,170.91	1,059.64	4,722.60	25,453.10		
1997	22,291.33	1,823.63	4,920.10	29,035.00		
1998				50,997.00		
1999				58,265.00		
2000				63,009.00		
2001				69,207.30	918.87	
2002				101,318.49	2,191.57	2,455.29
2003				142,306.92	3,779.40	3,444.08
2004				183,583.27	8,422.26	4,151.29
2005				213,194.58	17,096.14	5,553.97
2006				285,728.14	34,422.80	7,178.48
2007				382,302.71	62,173.20	9,995.71
2008				403,072.81	38,563.82	8,733.03
2009				426,749.60	30,428.11	6,771.58

Financial security

$$\boxed{\textbf{Financial Security} = \textbf{Cash Flow}}$$

Example (Interest Rate Swap)



Pricing problem: compute “fair” value of the security **today**.

Classification of financial securities

We classify all financial securities into 2 groups:

1. **Traded securities**: the price is *given* by the market.

$$\boxed{\textbf{Financial model} = \textbf{All traded securities}}$$

2. **Non-traded securities**: the price has to be *computed*.

Remark

This "black-and-white" classification is quite idealistic. Real life securities are usually "gray".

In this lecture we shall deal with **Arbitrage-Free Pricing**.

Arbitrage-free price

Inputs:

1. Financial model (collection of all traded securities)
2. A non-traded security.

Arbitrage strategy (intuitive definition):

1. start with zero capital (*nothing*)
2. end with positive and non zero wealth (*something*)

Assumption

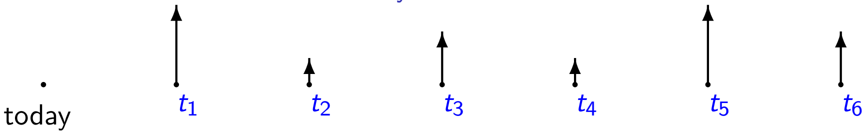
The financial model is arbitrage free.

Definition

An amount p is called an **arbitrage-free price** if, given an opportunity to trade the non-traded security at p, one is not able to construct an arbitrage strategy.

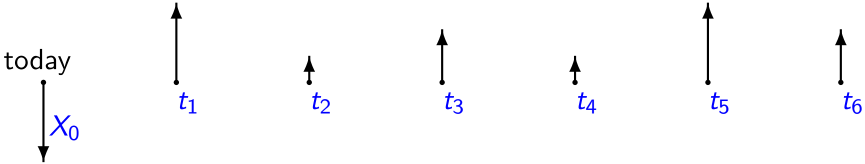
Replication

Cash flow of non-traded security:



Replicating strategy:

1. starts with some initial capital X_0
2. generates *exactly* the same cash flow in the future



Methodology of arbitrage-free pricing

Theorem

An arbitrage-free price p is unique if and only if there is a replicating strategy. In this case,

$$p = X_0,$$

where X_0 is the initial capital of a replicating strategy.

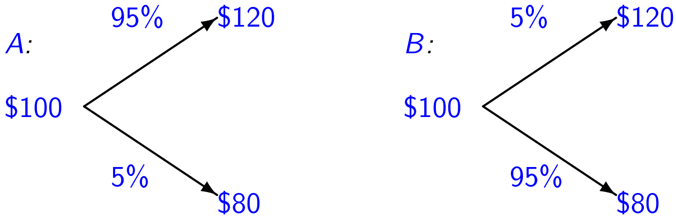
Main Principle:

$$\boxed{\textbf{(Unique) Arbitrage-Free Pricing} = \textbf{Replication}}$$

Problem on two calls

Problem

Consider two stocks: A and B. Assume that



Consider call options on A and B with the same strike $K = \$100$.
Assume that $T = 1$ and $r = 5\%$.
Compute the difference $C^A - C^B$ of their arbitrage-free prices.

Pricing in Black and Scholes model

There are two traded assets: a savings account and *a stock*.
We assume that the interest rate is zero:

$$r = 0.$$

The price of the stock:

$$dS_t = S_t \left(\mu dt + \sigma dW_t\right).$$

Here $W = (W_t)_{t \geq 0}$ is a Wiener process and
$\mu \in \mathbf{R}$: drift
$\sigma > 0$: volatility

Problem (Black and Scholes, 1973)

Compute arbitrage-free price V_0 of European put option with maturity T and payoff

$$\Psi = \max(K - S_T, 0).$$

Replication in Black and Scholes model

Basic principle: $\boxed{\textbf{Pricing} = \textbf{Replication}}$

Replicating strategy:

1. has wealth evolution:

$$X_t = X_0 + \int_0^t \Delta_u dS_u,$$

   where X_0 is the initial capital and Δ_t is the number of shares at time t;

2. generates *exactly* the same payoff as the option:

$$X_T(\omega) = \Psi(\omega) = \max(K - S_T(\omega), 0), \quad \mathbb{P}\text{-a.s.}.$$

Two standard methods: "direct" (PDE) and "dual" (martingales).

PDE method

Since $X_T = f(S_T)$ we look for replicating strategy in the form:

$$X_t = v(S_t, t)$$

for some deterministic $v = v(s, t)$. By Ito's formula,

$$dX_t = v_s(S_t, t)dS_t + (v_t(S_t, t) + \frac{1}{2}\sigma^2 S_t^2 v_{ss}(S_t, t))dt.$$

But, (since X is a wealth process)

$$dX_t = \Delta_t dS_t.$$

Hence, $v = v(s, t)$ solves PDE:

$$\begin{cases} v_t(s,t) + \frac{1}{2}\sigma^2 s^2 v_{ss}(s,t) & = & 0 \\ v(s,T) & = & \max(K - s, 0) \end{cases}$$

Martingale method

Observation: replication problem is defined “almost surely” and, hence, is invariant with respect to an equivalent change of probability measure.

Convenient choice: **martingale measure** $\mathbb{Q}$ for S. We have

$$dS_t = S_t \sigma dW_t^{\mathbb{Q}},$$

where $W^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$.

Replication strategy: (by Martingale Representation Theorem)

$$X_t = X_0 + \int_0^t \Delta dS = \mathbb{E}^{\mathbb{Q}}[\Psi | \mathcal{F}_t].$$

Risk-neutral valuation: (no replication!)

$$V_0 = \mathbb{E}^{\mathbb{Q}}[\Psi].$$

Martingale method

The computation of hedging delta is conveniently done with *Clark-Ocone formula*:

$$\sigma S_t \Delta_t = \mathbb{E}^{\mathbb{Q}}[\mathbf{D}_t^{\mathbb{Q}}[\psi]|\mathcal{F}_t],$$

where $\mathbf{D}^{\mathbb{Q}}$ is the *Malliavin derivative* under $\mathbb{Q}$. For example, for European put

$$\mathbf{D}_t^{\mathbb{Q}}[\max(K - S_T, 0)] = -1_{\{S_T<K\}}\mathbf{D}_t^{\mathbb{Q}}[S_T] = -1_{\{S_T<K\}}\sigma S_T,$$

resulting in

$$\Delta_t = -\frac{1}{S_t}\mathbb{E}^{\mathbb{Q}}[1_{\{S_T<K\}}S_T]|\mathcal{F}_t] = -\widetilde{\mathbb{Q}}[S_T < K|\mathcal{F}_t],$$

where

$$\frac{d\widetilde{\mathbb{Q}}}{d\mathbb{Q}} = \frac{S_T}{S_0}.$$

Arbitrage-free pricing: general financial model

There are $d+1$ *traded* or *liquid* assets:

1. a *savings account* with zero interest rate.
2. d *stocks*. The price process S of the stocks is a *semimartingale* on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$.

Question

Is the model **arbitrage-free**?

Question

Is the model **complete**? In other words, does it allow replication of any non-traded derivative?

Fundamental Theorems of Asset Pricing

Let $\mathcal{Q}$ denote the family of **martingale measures** for S, that is,

$$\mathcal{Q} = \{\mathbb{Q} \sim \mathbb{P} : \quad S \text{ is a local martingale under } \mathbb{Q}\}$$

Theorem (**1st FTAP**)

$$\textbf{Absence of arbitrage} \quad \Longleftrightarrow \quad \mathcal{Q} \neq \emptyset.$$

Theorem (**2nd FTAP**)

$$\textbf{Completeness} \quad \Longleftrightarrow \quad |\mathcal{Q}| = 1.$$

Risk-Neutral Valuation

Consider a European option with payoff Ψ at maturity T. The formula

$$V_0 = \mathbb{E}^{\mathbb{Q}}[\Psi],$$

where $\mathbb{Q} \in \mathcal{Q}$ is called **Risk-Neutral Valuation**.

Arbitrage-free models:

$$\boxed{\textbf{Unique Arbitrage-Free Pricing = Replication}}$$

Complete models: (no replication!)

$$\boxed{\textbf{Arbitrage-Free Pricing = Risk-Neutral Valuation}}$$

Free Lunch with Vanishing Risk

For 1st FTAP to hold true the following definition of arbitrage is needed (Delbaen and Schachermayer, Math Ann, 1994):

1. There is a set $A \in \Omega$ with $\mathbb{P}[A] > 0$.
2. For any $\epsilon > 0$ there is a strategy X such that
   2.1 X is *admissible*, that is, for some constant $c > 0$,
   $$X \geq -c.$$
   2.2 $X_0 \leq \epsilon$ (start with almost nothing)
   2.3 $X_T \geq 1_A$ (end with something)

Verification of the absence of arbitrage

Assume that $\mathcal{F}_t = \mathcal{F}_t^S$ (the information is generated by S). Then without loss in generality $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$ is a canonical probability space of continuous functions $\omega = \omega(t)$ on $[0, T]$ and $S_t(\omega) = \omega(t)$. Suppose

$$S_t = S_0 + \int_0^t \mu_t du + W_t^{\mathbb{P}},$$

where $\mu_t = \mu((S_u)_{u \le t}, t)$ and $W^{\mathbb{P}}$ is a $\mathbb{P}$-Brownian motion.

Problem

Find (necessary and sufficient) conditions on $\mu = (\mu_t)$ for the absence of arbitrage (No FLVR).

Solution

Levi's theorem $\implies$ that the only possible martingale measure $\mathbb{Q}$ is such that

$$W_t^{\mathbb{Q}} = S_t - S_0,$$

is a $\mathbb{Q}$-Brownian motion. Then by 1st FTAP

$$\text{No FLVR} \quad \iff \quad \mathbb{P} \sim \mathbb{Q}.$$

One can show (easy!) that

$$\mathbb{P} \sim \mathbb{Q} \quad \iff \quad \int_0^T \mu_t^2 dt < \infty \quad \mathbb{P} + \mathbb{Q} \quad \text{a.s.}.$$

Martingale Representation (MR)

$(\Omega, \mathcal{F}_T, \mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$: a complete filtered probability space.

$\mathbb{Q}$: an equivalent probability measure.

$S = (S_t^j)$: J-dimensional martingale under $\mathbb{Q}$.

Martingale representation (MR= MR(S)): every martingale $M = (M_t)$ under $\mathbb{Q}$ admits an integral representation with respect to S, that is,

$$M_t = M_0 + \int_0^t H_u dS_u, \quad t \in [0, T],$$

for some predictable S-integrable process $H = (H_t^j)$.

- Completeness in Mathematical Finance.
- Jacod's Theorem (2nd FTAP): MR holds iff $\mathbb{Q}$ is the only martingale measure for S.

Brownian framework

Hereafter, we assume that

$$\mathbb{P} = \mathbb{Q}$$

and that there is a d-dimensional Brownian motion W such that

$$\mathcal{F}_t = \mathcal{F}_t^W, \quad t \in [0, T].$$

Ito's MR(W) : every martingale $M = (M_t)$ admits an integral representation

$$M_t = M_0 + \int_0^t H_u dW_u, \quad t \in [0, T],$$

for some adapted process $H = (H_t)$ such that

$$\int_0^T |H|_t^2 dt < \infty.$$

Forward setup

Input: *volatility* process $\Sigma = (\Sigma_t)$ taking values in $J \times d$-matrices.

Stocks prices $S = (S^j)$ are a J-dimensional Ito's martingale:

$$S_t = S_0 + \int_0^t \Sigma_u dW_u$$

By Ito's MR(W) we have that MR(S) holds if and only if for every H there is K such that

$$\int HdW = \int KdS = \int K\Sigma dW$$

which, by linear algebra, is the case if and only if

$$\mathrm{rank}(\Sigma) = d, \quad (d\mathbb{P} \times dt - a.s.).$$

- Very easy to verify!

Backward setup

Input: terminal values $\psi = (\psi^j)$ for S:

$$S_t = \mathbb{E}[\psi|\mathcal{F}_t], \quad t \in [0, T].$$

- This setup arises, e.g, when liquid securities are derivatives: forwards, futures, options; also, in equilibrium.
- Ito's MR(W) $\implies$ the existence of $\Sigma = (\Sigma_t)$ such that

$$S_t = S_0 + \int_0^t \Sigma_u dW_u$$

- Conditions on ψ so that MR(S) holds $\Leftrightarrow$ $\text{rank}(\Sigma) = d$?

Summary

- (FTAP1): Model is *arbitrage-free*
  $\Leftrightarrow$ There is a martingale measure $\mathbb{Q}$ for prices S.
- (FTAP2): Model is *complete*
  $\Leftrightarrow$ There is only one martingale measure $\mathbb{Q}$ for S.
  $\Leftrightarrow$ *MR* property holds for S under $\mathbb{Q}$.
  - Easy to verify in forward setup.
  - Non-trivial research topic in backward framework.
- Unique Arbitrage-Free Pricing $=$ Replication.
- Replication is hard. In *complete* models we can avoid replication and compute prices as expectations $\mathbb{E}^{\mathbb{Q}}[\Psi]$.