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The mathematics of financial risk management: approach, challenges, and recent developments

by Walter Farkas
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Capital adequacy

As scientists we observe the real world and build mathematical models of it. This allows us to (approximately) describe the world and solve problems therein. One such problem evolves around capital adequacy:

- Liability holders and regulators of financial institutions are credit sensitive.
- They are concerned that the value of the institution's **assets** is insufficient to cover its **liabilities**.
- To address this concern financial institutions hold **risk capital**, which is meant to absorb unexpected losses.

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The capital adequacy problem:

- ▷ **How much risk capital** should a financial institution hold to be deemed **adequately capitalized** by the regulator?

Related regulatory frameworks:
Swiss Solvency Test, Solvency II, Basel III-IV.

Objective of the presentation

- ▷ Highlight mathematical aspects of the capital adequacy problem and examine a new form of it.
- ▷ Introduce recently developed risk measures that capture such situations in the original spirit of Artzner, Delbaen, Eber, Heath [ADEH99].
- ▷ Investigate possible acceptability adjustment procedures based on monetary risk measures and a new concept: intrinsic risk measures.

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Co-authors and acknowledgements

The presentation is based on the following papers

- W. Farkas, P. Koch-Medina, C. Munari:
  Capital requirements with defaultable securities,
  Insurance: Mathematics and Economics, 55, 58-67, (2014)
- W. Farkas, P. Koch-Medina, C. Munari:
  Beyond cash-additive capital requirements: when discounting fails,
  Finance and Stochastics, 18 (1), 145-173, (2014)
- W. Farkas, A. Smirnow:
  Intrinsic risk measures,
  Submitted, 2016 (SSRN)

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Formalizing the capital adequacy problem

Key ingredients are:

- a set $\mathcal{X}$ representing net **terminal financial positions**,
- an acceptance set $\mathcal{A} \subset \mathcal{X}$ representing **acceptable positions**,
- a class M of **admissible management actions**,
- a **cost function** $c : M \to \mathbb{R}$,
- an **impact function** $I : \mathcal{X} \times M \to \mathcal{X}$.

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The capital required to make an unacceptable position $X \in \mathcal{X}$ acceptable by implementing an admissible management action can be defined as

$$\rho(X) = \inf\{c(m) \in \mathbb{R} \,|\, m \in M : I(X, m) \in \mathcal{A}\}.$$

Key observation is:

▷ the acceptance set $\mathcal{A}$ is the only pre-specified element (by regulator).

Terminology and preliminaries

In *mathematical finance*, we often use the notion of probability spaces and random variables.

- Let $(\Omega, \mathcal{F})$ be a **measurable space** defined by a sample space $\Omega \neq \emptyset$ and a σ-algebra $\mathcal{F} \subset \mathcal{P}(\Omega)$.
- **Real-valued random variables** $X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ serve as representatives of future **financial positions**. Let $\mathcal{L}^0(\Omega, \mathcal{F})$ denote the vector space of all such $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable functions.
- Adding a **probability measure** $\mathbb{P} : \mathcal{F} \to [0, 1]$ to the tuple $(\Omega, \mathcal{F})$ we get a **probability space** $(\Omega, \mathcal{F}, \mathbb{P})$.
- Denote by $L^0(\Omega, \mathcal{F}, \mathbb{P})$ the vector space of equivalence classes in $\mathcal{L}^0(\Omega, \mathcal{F})$ with respect to $\mathbb{P}$-almost sure equality and equip it with the topology of convergence in probability.

Financial positions

In general, given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, financial positions live in

- topological (we can determine if positions are close to each other)
- vector (we can aggregate positions)

  spaces

- equipped with the $\mathbb{P}$-almost sure ordering:
  $X \leq Y$ if and only if $\mathbb{P}[X \leq Y] = 1$.

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Example Possible choices could be

- $L^\infty(\Omega, \mathcal{F}, \mathbb{P}) = \{X \in L^0(\Omega, \mathcal{F}, \mathbb{P}) \mid \|X\|_{L^\infty(\mathbb{P})} < \infty\}$ equipped with the *supremum norm* $\|X\|_{L^\infty(\mathbb{P})} = \mathbb{P}\text{-}\operatorname{ess\,sup}_{\omega\in\Omega} |X(\omega)|$.
- For $p \in [1, \infty)$, $L^p(\Omega, \mathcal{F}, \mathbb{P}) = \{X \in L^0(\Omega, \mathcal{F}, \mathbb{P}) \mid \|X\|_{L^p(\mathbb{P})} < \infty\}$ equipped with the L^p*-norm* $\|X\|_{L^p(\mathbb{P})} = \mathbb{E}[|X|^p]$.

Acceptable financial positions

Introduce acceptability structure to the mathematical framework.

Definition (Acceptance sets) A subset $\mathcal{A} \subset \mathcal{X}$ is called an **acceptance set** if it is

- *non-trivial*, i.e. $\mathcal{A} \neq \emptyset$ and $\mathcal{A} \subsetneq \mathcal{X}$, and
- *monotone*, i.e. $X_T \in \mathcal{A}$, $Y_T \in \mathcal{X}$, and $Y_T \geq X_T$ imply $Y_T \in \mathcal{A}$.

These assumptions are based on
minimal requirements of financial rationality:

▷ Some, but not all, positions should be acceptable,

▷ A position dominating an acceptable position should be acceptable.

Properties of acceptance sets

An acceptance set $\mathcal{A} \subset \mathcal{X}$ is called

- a *cone* if $X_T \in \mathcal{A} \implies \forall \lambda > 0 : \lambda X_T \in \mathcal{A}$,
- *convex* if $X_T, Y_T \in \mathcal{A} \implies \forall \lambda \in [0,1] : \lambda X_T + (1-\lambda) Y_T \in \mathcal{A}$,
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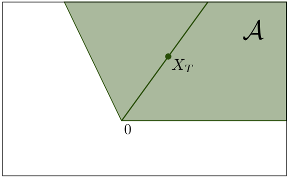

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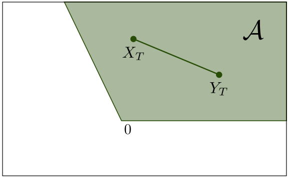

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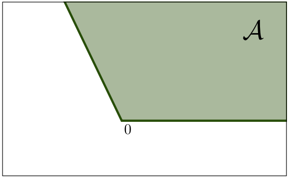

Examples of acceptance sets

Let $q_\alpha^+(X)$ be the upper quantile of a random variable X at level $\alpha \in (0, 1)$.

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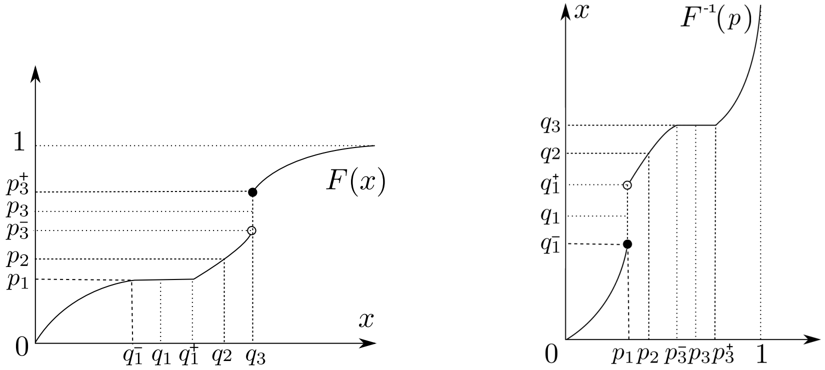

Figure: Distribution function F and its quantile function F^{-1}. Source: `https://en.wikipedia.org/wiki/Quantile_function`

Examples of acceptance sets

Let $q_\alpha^+(X)$ be the upper quantile of a random variable X at level $\alpha \in (0,1)$.

1. The **quantile-based** (or VaR_α) acceptance set is

$$\mathcal{A}_\alpha = \{X \in \mathcal{X} \,|\, q_\alpha^+(X) \geq 0\} = \{X \in \mathcal{X} \,|\, \mathbb{P}[X < 0] \leq \alpha\}.$$

   $\mathcal{A}_\alpha$ is a cone, but it is not convex in general.

2. The **shortfall quantile-based** (or ES_α) acceptance set at level $\alpha \in (0,1)$ is

$$\mathcal{A}^\alpha = \left\{X \in \mathcal{X} \,|\, \mathbb{E}\left[X \mathbf{1}_{\{X \leq q_\alpha^+(X)\}}\right] \geq 0\right\},$$

   assuming that X has a continuous distribution function (which implies $\mathbb{P}[X \leq q_\alpha^+(X)] = \alpha$). $\mathcal{A}^\alpha$ is coherent.

Investing in a single asset

A first specification to the capital adequacy problem.
Landmark reference Artzner, Delbaen, Eber, Heath [ADEH99].

Take a traded asset $S = (S_0, S_T)$ with current value $S_0 > 0$ and terminal nonzero payoff $S_T \geq 0$, and let

- $M = \{\lambda S_T \mid \lambda \in \mathbb{R}\}$, representing payoffs of positions in S,
- $c(\lambda S_T) = \lambda S_0$, representing the cost of an investment in S,
- $I(X, \lambda S_T) = X + \lambda S_T$, representing the impact on the position.

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Definition The **risk measure** or **capital requirement** based on $(\mathcal{X}, \mathcal{A}, M, c, I)$ as above is the map $\rho_{\mathcal{A},S} : \mathcal{X} \to \overline{\mathbb{R}}$ defined by

$$\begin{aligned}\rho_{\mathcal{A},S}(X) &= \inf\{c(\lambda S_T) \in \mathbb{R} \mid I(X, \lambda S_T) \in \mathcal{A}\} \\ &= \inf\{\lambda S_0 \in \mathbb{R} \mid X + \lambda S_T \in \mathcal{A}\} = \inf\left\{m \in \mathbb{R} \mid X + \frac{m}{S_0} S_T \in \mathcal{A}\right\}.\end{aligned}$$

General properties of risk measures

Proposition Let $\mathcal{A} \subset \mathcal{X}$ be an acceptance set, and S a traded asset with price $S_0 > 0$ and nonzero payoff $S_T \in \mathcal{X}_+$. Then

- $\rho_{\mathcal{A},S}$ is S-**additive**, that means for $\lambda \in \mathbb{R}$

$$\rho_{\mathcal{A},S}(X + \lambda S_T) = \rho_{\mathcal{A},S}(X) - \lambda S_0$$

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- $\rho_{\mathcal{A},S}$ is **decreasing**, since $\mathcal{A}$ is monotone, that means

$$X \leq Y \quad \text{implies} \quad \rho_{\mathcal{A},S}(X) \geq \rho_{\mathcal{A},S}(Y)\,.$$

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From a financial point of view

- S-additivity: investing in S has a linear effect on cap. requirements.
- Monotonicity: riskier positions need higher risk capital.

Cash-additive risk measures

Main references
Föllmer, Schied [FS04], Frittelli, Rosazza Gianin [FRG02].

Definition Let $\mathcal{A} \subset \mathcal{X}$ be an acceptance set, and $S = (1, \mathbf{1}_\Omega)$. The risk measure $\rho_{\mathcal{A},S}$ is called **cash-additive** and we write

$$\rho_{\mathcal{A}}(X) = \rho_{\mathcal{A},1}(X) = \inf\{m \in \mathbb{R} \,|\, X + m\mathbf{1}_\Omega \in \mathcal{A}\}\,.$$

In this case, S-additivity is called **cash-additivity**, meaning for $\lambda \in \mathbb{R}$

$$\rho_{\mathcal{A}}(X + \lambda\mathbf{1}_\Omega) = \rho_{\mathcal{A}}(X) - \lambda\,.$$

From a financial point of view

- the traded asset corresponding to a cash-additive risk measure can be regarded as a **risk-free bond** with zero interest rate.

Why cash-additive? Choice of the numéraire

Typical argument in favour of cash-additive risk measures:

Let $\mathcal{A} \subset \mathcal{X}$ be an acceptance set and $S = (S_0, rS_0)$ a bond with return rate $r > 0$. The risk measure $\rho_{\mathcal{A},S}$ can be expressed in terms of $\rho_{\mathcal{A}_S}$ as

$$\rho_{\mathcal{A},S}(X) = S_0\, \rho_{\mathcal{A}_S}\left(\frac{X}{S_T}\right), \quad \text{where } \mathcal{A}_S = \left\{\frac{X}{S_T} \,\middle|\, X \in \mathcal{A}\right\}.$$

From an accounting point of view

- X is a future position expressed in cash,
- $\rho_{\mathcal{A},S}(X)$ is a capital amount today,
- $\frac{X}{S_T}$ is a future **discounted** position,
- $\rho_{\mathcal{A}_S}\left(\frac{X}{S_T}\right)$ is a capital amount today corresponding to a future **discounted** position.

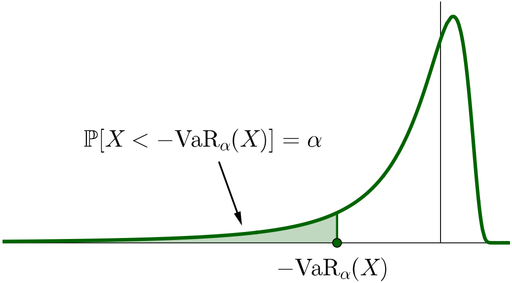
Examples of cash-additive risk measures I

The **Value-at-Risk** of a position $X \in \mathcal{X}$ at level $\alpha \in (0,1)$ is

$$\mathrm{VaR}_\alpha(X) = \rho_{\mathcal{A}_\alpha}(X) = -q^+_\alpha(X)\,,$$

where

$$\mathcal{A}_\alpha = \{X \in \mathcal{X} \,|\, \mathbb{P}[X < 0] \le \alpha\}\,.$$



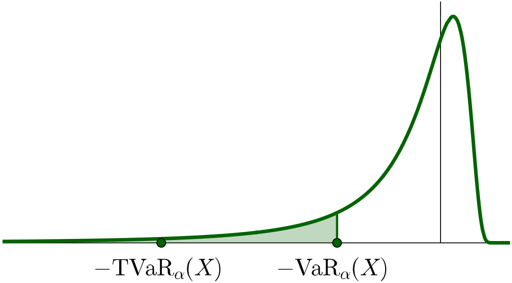
Examples of cash-additive risk measures II

The **Tail Value-at-Risk** of a position $X \in \mathcal{X}$ at level $\alpha \in (0,1)$ is

$$\mathrm{TVaR}_\alpha(X) = \rho_{\mathcal{A}^\alpha}(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)\mathrm{d}\beta\,,$$

where

$$\mathcal{A}^\alpha = \{X \in \mathcal{X} \,|\, \mathrm{TVaR}_\alpha(X) \le 0\}\,.$$



Cash additivity: reduction or simplification?

The cash-additive reduction should translate the original mathematical problem into a more tractable one.

- In this respect, the key financial question is:
  Can every relevant financial problem for the original model be formulated after the cash-additive reduction?
- The complementary key mathematical question is:
  Does every relevant mathematical property of $\rho_{\mathcal{A},S}$ admit a counterpart in the reduced cash-additive setting?

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Unless the reference asset is a risk-free bond, neither question has a positive answer!

One cannot always count on discounting

Let $\mathcal{A}$ be an acceptance set and $S = (S_0, S_T)$ a traded asset with non-zero payoff $S_T \in \mathcal{X}_+$, and recall that

$$\rho_{\mathcal{A},S}(X) = \inf \left\{ m \in \mathbb{R} \,\middle|\, X + \frac{m}{S_0} S_T \in \mathcal{A} \right\}.$$

One question

Can we reduce $\rho_{\mathcal{A},S}$ to a cash-additive risk measure?

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Two comments

- If S_T is *bounded away from zero*, that is $S_T \geq \epsilon$ for some $\epsilon > 0$, then discounting works.
- If not, either we lose control over the space where X/S_T belongs to, or we cannot even define X/S_T.

One cannot always count on discounting

Two situations where the reduction does not work:

▷ Let $S_T \sim \log \mathcal{N}(\mu, \sigma)$ be **log-normally distributed**.

- Since the distribution is continuous, we have $\mathbb{P}[S_T = 0] = 0$.
- But $\mathbb{P}[S_T \leq \lambda] = \frac{1}{2} + \frac{1}{2}\mathrm{erf}(\frac{\log \lambda - \mu}{\sqrt{2}\sigma}) > 0$ for all $\lambda > 0$.
- Hence, S_T is not bounded away from zero.
- Discounting is not possible as we lose control of the space of discounted positions.

▷ Let S_T be any **defaultable instrument**, that is $\mathbb{P}[S_T = 0] > 0$. Discounting is not possible as we cannot define a discounted space.

Original approach

Reintroduce maps $\rho_{\mathcal{A},S} : \mathcal{X} \to \overline{\mathbb{R}}$ of the form

$$\rho_{\mathcal{A},S}(X) = \inf \left\{ m \in \mathbb{R} \,|\, X + \frac{m}{S_0} S_T \in \mathcal{A} \right\} .$$

Investigation of the properties of $\rho_{\mathcal{A},S}$ without preliminary assumptions on $\mathcal{A}$ and S. In particular,

- S_T might be not bounded away from zero, allowing **default profiles**.
- S_T might even be zero in some future scenarios, allowing **extreme default profiles**, e.g. **zero recovery** within time $t = T$.

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The main advantage of this approach is the possibility to fix an acceptability criterium $\mathcal{A}$ and get a **family of risk measures compatible with** $\mathcal{A}$ by simply changing assets S.

Finiteness and continuity properties of $\rho_{\mathcal{A},S}$

Requiring $\rho_{\mathcal{A},S}$ to be **finite** and **continuous** at $X \in \mathcal{X}$ is economically meaningful:

- If $\rho_{\mathcal{A},S}(X) = \infty$, then X cannot be made acceptable by investing any amount of capital in the asset S, suggesting S is not a good vehicle to reach acceptability.
- If $\rho_{\mathcal{A},S}(X) = -\infty$, then extraction of arbitrary amounts of capital without loosing acceptability is possible, suggesting that $\mathcal{A}$ might be too large.
- If $\rho_{\mathcal{A},S}$ is not continuous at X, then a slight change in the balance sheet may lead to a dramatical change in the corresponding required capital.
- (Lower semi-) continuity typically allows to obtain important dual representations.

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The interplay between $\mathcal{A}$ and S

Example: Value-at-Risk acceptability

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be nonatomic, and set $\mathcal{X} = L^p$ with $0 \leq p \leq \infty$.
Recall that for $0 < \alpha < 1$

$$\mathcal{A}_\alpha = \{X \in L^p \,|\, \mathbb{P}[X < 0] \leq \alpha\},$$

$$\mathrm{VaR}_\alpha(X) = \rho_{\mathcal{A}_\alpha}(X) = \inf\{m \in \mathbb{R} \,|\, \mathbb{P}[X + m < 0] \leq \alpha\}.$$

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$$\mathcal{A}_\alpha = \{X \in L^p \,|\, \mathbb{P}[X < 0] \leq \alpha\},$$

$$\mathrm{VaR}_\alpha(X) = \rho_{\mathcal{A}_\alpha}(X) = \inf\{m \in \mathbb{R} \,|\, \mathbb{P}[X + m < 0] \leq \alpha\}.$$

Proposition Let S be the reference asset.

1. Assume $p = \infty$. Then the following statements hold:
    - $\rho_{\mathcal{A}_\alpha, S}$ is finite if and only if $\mathrm{VaR}_\alpha(S_T) < 0 < \mathrm{VaR}_\alpha(-S_T)$,
    - $\rho_{\mathcal{A}_\alpha, S}$ is continuous if and only if $S_T \geq \epsilon$ $\mathbb{P}-\mathrm{a.s.}$ for some $\epsilon > 0$.

The interplay between $\mathcal{A}$ and S

Example: Value-at-Risk acceptability

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2. Assume $p < \infty$. Then the following statements hold:
   - $\rho_{\mathcal{A}_\alpha, S}$ is finite if and only if $\mathbb{P}[S_T = 0] < \min\{\alpha, 1 - \alpha\}$,
   - $\rho_{\mathcal{A}_\alpha, S}$ is never continuous on whole L^p.

The interplay between $\mathcal{A}$ and S

Example: Tail Value-at-Risk acceptability

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be nonatomic, and set $\mathcal{X} = L^p$ with $1 \leq p \leq \infty$.
Recall that for $0 < \alpha < 1$

$$\mathrm{TVaR}_\alpha(X) = \rho_{\mathcal{A}^\alpha}(X) = \frac{1}{\alpha}\int_0^\alpha \mathrm{VaR}_\beta(X)\,\mathrm{d}\beta\,.$$

The interplay between $\mathcal{A}$ and S

Example: Tail Value-at-Risk acceptability

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Proposition Let S be the reference asset.
The following statements are equivalent:

1. $\rho_{\mathcal{A}^\alpha,S}$ is finitely valued,
2. $\rho_{\mathcal{A}^\alpha,S}$ is (Lipschitz) continuous,
3. there exists a $\lambda > 0$ such that $\mathbb{P}[S_T < \lambda] < \alpha$,
4. $\mathrm{TVaR}_\alpha(S_T) < 0$.

Intermediate summary

The capital requirement of a position X based on $\mathcal{A}$ and $S = (S_0, S_T)$ has been defined as

$$\rho_{\mathcal{A},S}(X) = \inf\left\{m \in \mathbb{R} \mid X + \frac{m}{S_0} S_T \in \mathcal{A}\right\}.$$

Main results

- ▷ Extension of the theory of cash-additive risk measures.
- ▷ The asset S is not assumed to be **risk-free**: S might describe a **stock**, a zero-coupon bond with **stochastic recovery**, a general **defaultable security**.
- ▷ Finiteness and continuity as a result of the **interplay** between $\mathcal{A}$ and S.

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Recent developments: intrinsic risk measures

We have learned

Monetary risk measures suggest a procedure to make an unacceptable position acceptable:

> *Given an unacceptable position X, $\rho_{\mathcal{A},S}(X)/S_0$ units of the reference asset S are bought and added to make the overall position acceptable.*

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However, one must raise and carry the monetary amount $\rho_{\mathcal{A},S}(X)$ suggested by the risk measure.

- ▷ This can be difficult in practice.
- ▷ Monetary risk measures do not entirely account for the acquisition of capital.

Recent developments: intrinsic risk measures

Monetary risk measures are based on the following suggestion in [ADEH99, Section 2.1]:

> *'The current cost of getting enough of this or these [commonly accepted] instrument(s) is a good candidate for a measure of risk of the initially unacceptable position.'*

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Monetary risk measures are based on the following suggestion in [ADEH99, Section 2.1]:

> *'The current cost of getting enough of this or these [commonly accepted] instrument(s) is a good candidate for a measure of risk of the initially unacceptable position.'*

This is not the only applicable approach and they also mention that

> *'For an unacceptable risk [...] one remedy may be to alter the position.'*

Introducing intrinsic risk measures

We propose to use the current value of the financial position as a specified amount of available capital, and invest it into a reference asset.

Let $\mathcal{A} \subset \mathcal{X}$ be a closed acceptance set containing 0.

Definition (Extending the notion of financial positions)

- Financial positions are tuples $X = (X_0, X_T) \in \mathbb{R}_{>0} \times \mathcal{X}$.
- $S = (S_0, S_T)$ is an eligible asset if $S \in \mathbb{R}_{>0} \times \mathcal{A}$ and $S_T \geq 0$.

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Definition (Intrinsic risk measure) Let S be an eligible asset. An *intrinsic risk measure* is a map $R_{\mathcal{A},S} : \mathbb{R}_{>0} \times \mathcal{X} \to [0,1]$ defined by

$$R_{\mathcal{A},S}(X) = \inf \left\{ \lambda \in [0,1] \,\middle|\, (1-\lambda) X_T + \lambda \frac{X_0}{S_0} S_T \in \mathcal{A} \right\}.$$

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Motivation

- Dissociate from hypothetical money as a measure of risk, and reflect on the initial value of your financial position.
- Direct path towards the acceptance set leads to less costly management actions.
- Treatment of risk measures tending to infinity becomes redundant, and the focus lies on unacceptable positions.
- Monotonicity and quasi-convexity follow from the structure of the acceptance set.

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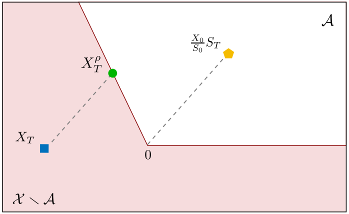
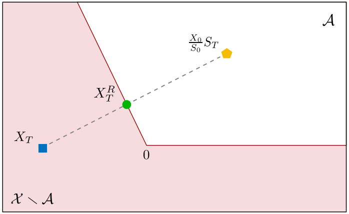
Illustration of the new approach

- Intrinsic risk measures are directly based on the diversification principle and provide more direct paths towards the acceptance set.



(a) Monetary risk measures



(b) Intrinsic risk measures

Figure: The payoff of the eligible asset (yellow ⬠) is used to make the unacceptable position (blue □) acceptable (green ○).

Conic acceptance sets, e.g. VaR and TVaR acceptability

Let $\mathcal{A}$ be a closed, conic acceptance set and recall the monetary risk measure

$$\rho_{\mathcal{A},S}(X_T) = \inf\left\{ m \in \mathbb{R} \,\middle|\, X_T + \frac{m}{S_0} S_T \in \mathcal{A} \right\}.$$

The intrinsic risk measure with respect to the same acceptance set and eligible asset can be written as

$$R_{\mathcal{A},S}(X) = \frac{(\rho_{\mathcal{A},S}(X_T))^+}{X_0 + \rho_{\mathcal{A},S}(X_T)}.$$

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Consequences

- The intrinsic approach requires less nominal value to reach acceptability (holds also true for convex acceptance sets), this means

$$X_0 R_{\mathcal{A},S}(X) \leq \rho_{\mathcal{A},S}(X_T)\,.$$

- The intrinsic approach yields financial positions with the same performance as those resulting from the traditional approach.

$$\text{Traditional approach: } X_0 \longmapsto X_0^{\rho} = X_0 + \rho_{\mathcal{A},S}(X_T)\,,$$

$$X_T \longmapsto X_T^{\rho} = X_T + \frac{\rho_{\mathcal{A},S}(X_T)}{S_0} S_T\,.$$

$$\text{Intrinsic approach: } X_0 \longmapsto X_0^{R} = X_0\,,$$

$$X_T \longmapsto X_T^{R} = (1 - R_{\mathcal{A},S}(X))X_T + \frac{X_0 R_{\mathcal{A},S}(X)}{S_0} S_T\,.$$

On cones, we have

$$\frac{X_T^{\rho}}{X_0^{\rho}} = \frac{X_T^{R}}{X_0^{R}}\,.$$

Convex acceptance sets, e.g. TVaR and utility acceptance

On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ let

- $\mathcal{A}$ be a closed, convex acceptance set,
- $\mathcal{M}_\sigma(\mathbb{P})$ be the set of all σ-additive probability measures $\mathbb{Q} \ll \mathbb{P}$, and
- $\alpha(\mathbb{Q}, \mathcal{A}) = \inf_{Y_T \in \mathcal{A}} \mathbb{E}_\mathbb{Q}[Y_T]$ be a **penalty function**.

Intrinsic risk measures can be written as

$$R_{\mathcal{A},S}(X) = \sup_{\mathbb{Q} \in \mathcal{M}_\sigma(\mathbb{P})} \frac{(\alpha(\mathbb{Q}, \mathcal{A}) - \mathbb{E}_\mathbb{Q}[X_T])^+}{\frac{X_0}{S_0}\mathbb{E}_\mathbb{Q}[S_T] - \mathbb{E}_\mathbb{Q}[X_T]} .$$

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Consequences

- Intrinsic risk measures have a dual representation: a normalized version of the dual representation of monetary risk measures.

Thank you
for your attention!

References I

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Dimitrios G. Konstantinides and Christos E. Kountzakis. Risk measures in ordered normed linear spaces with non-empty cone-interior. *Insurance: Mathematics and Economics*, 48(1):111–122, 2011.

Literature outside cash-additivity

Risk measures of the form $\rho_{\mathcal{A},S}$ have been occasionally treated.

- ▷ Artzner, Delbaen, Eber, and Heath [ADEH99] work on a finite state space under the assumption that S_T is bounded away from zero.
- ▷ Frittelli and Scandolo [FS06] provide a result on finiteness on L^∞ under the assumption that S_T is bounded away from zero.
- ▷ Filipović and Kupper [FK08] show that if $\mathcal{X}$ is an ordered normed space and there exists $\lambda > 0$ such that $X \geq -\lambda \|X\| S_T$ for every $X \in \mathcal{X}$, then $\rho_{\mathcal{A},S}$ is finitely valued and Lipschitz continuous. The previous assumption is equivalent to $S_T \in \mathrm{int}(\mathcal{X}_+)$.
- ▷ Artzner, Delbaen, and Koch-Medina [ADKM09] work on a finite sample space under the assumption that S_T is bounded away from zero.
- ▷ Konstantinides and Kountzakis [KK11] show continuity properties of $\rho_{\mathcal{A},S}$ on an ordered normed space $\mathcal{X}$ under the assumption that $S_T \in \mathrm{int}(\mathcal{X}_+)$.