Dynamic Blocking Problems for a Model of Fire Propagation

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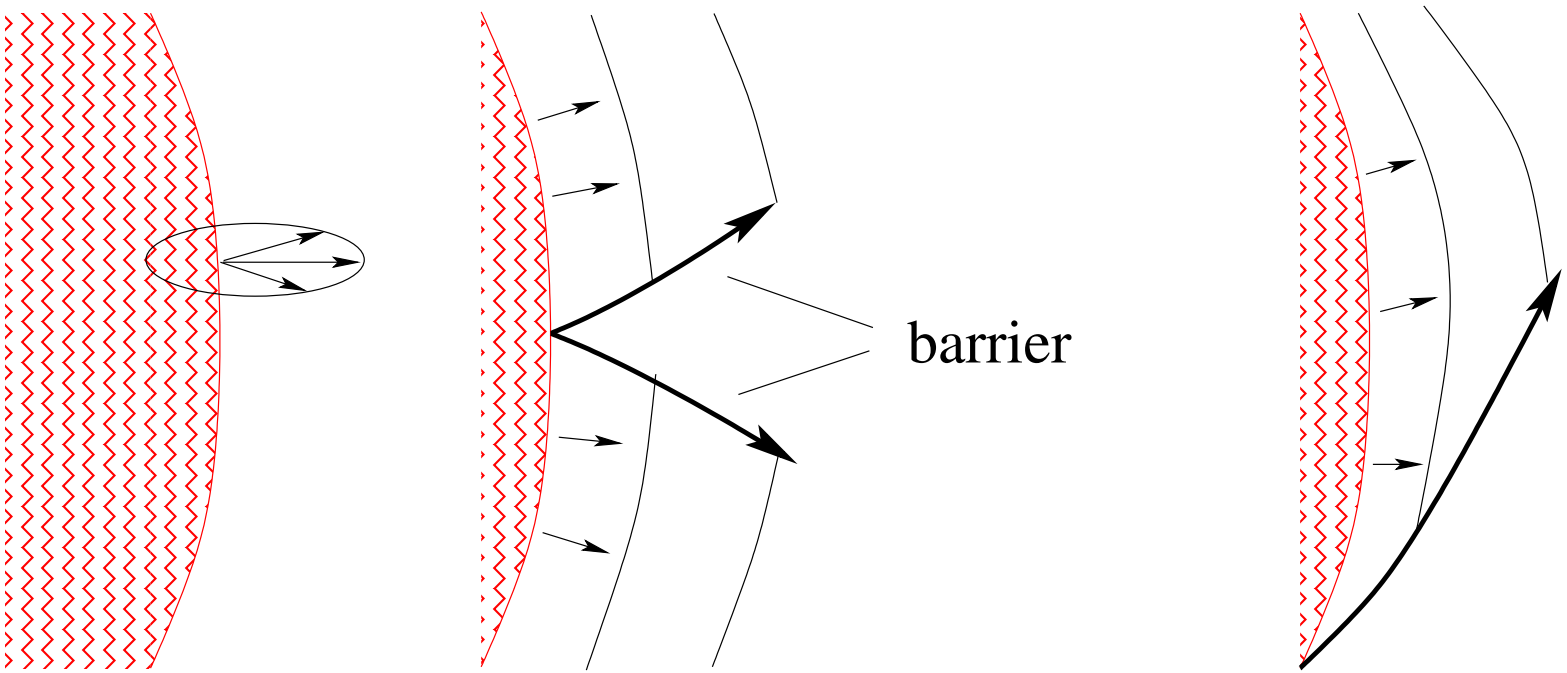
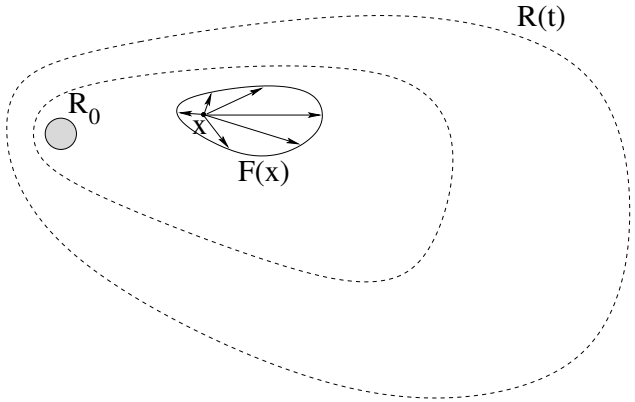
(Waterloo, 2011)

Blocking an advancing wildfire



A Differential Inclusion Model for Fire Propagation



$$R(t) \subset \mathbb{R}^2 = \text{set reached by the fire at time } t \geq 0$$

determined as the reachable set by a **differential inclusion**

$$\dot{x} \in F(x) \qquad\qquad x(0) \in R_0 \subset \mathbb{R}^2$$

Fire may spread in different directions with different velocities

$$R(t) = \Big\{ x(t)\,; \quad x(\cdot) \ \text{ absolutely continuous}\,,$$

$$x(0) \in R_0\,, \quad \dot{x}(\tau) \in F\big(x(\tau)\big) \ \text{ for a.e. } \tau \in [0,t] \Big\}$$

Confinement Strategies

(A.B., J.Differential Equations, 2007)

Assume: a **controller** can construct a **wall**, i.e. a one-dimensional rectifiable curve γ, which blocks the spreading of the fire.

$$\gamma(t) \subset \mathbb{R}^2 = \text{portion of the wall constructed within time } t$$

$$\sigma = \text{speed at which the wall is constructed}$$

Definition 1. A set valued map $t \mapsto \gamma(t) \subset \mathbb{R}^2$ is an **admissible strategy** if :

(H1) For every $t_1 \leq t_2$ one has $\gamma(t_1) \subseteq \gamma(t_2)$

(H2) Each $\gamma(t)$ is a rectifiable set (possibly not connected). Its length satisfies

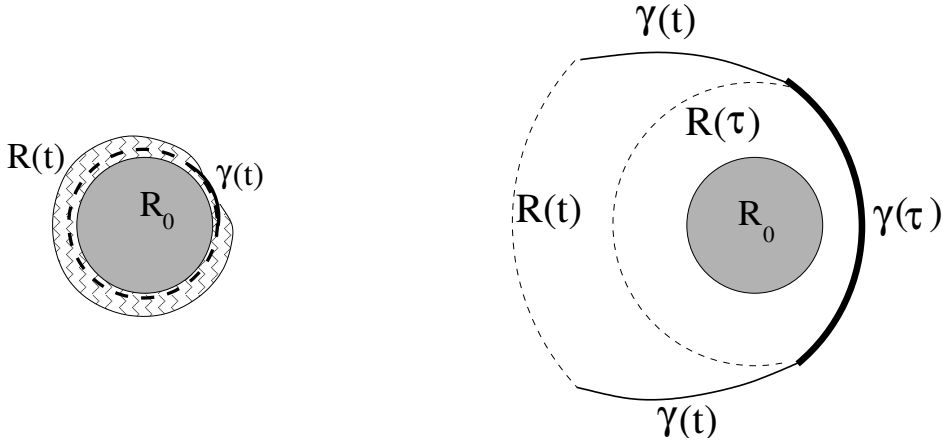
$$m_1(\gamma(t)) \;\leq\; \sigma\, t$$

Definition 2. The reachable set determined by the blocking strategy γ is

$$R^\gamma(t) \doteq \Big\{ x(t)\,; \quad x(\cdot) \ \text{ absolutely continuous}\,, \quad x(0) \in R_0$$

$$\dot{x}(\tau) \in F\big(x(\tau)\big) \ \text{ for a.e. } \tau \in [0,t]\,, \qquad x(\tau) \notin \gamma(\tau) \ \text{ for all } \tau \in [0,t] \Big\}$$

REMARK: Walls must be constructed in **real time !**

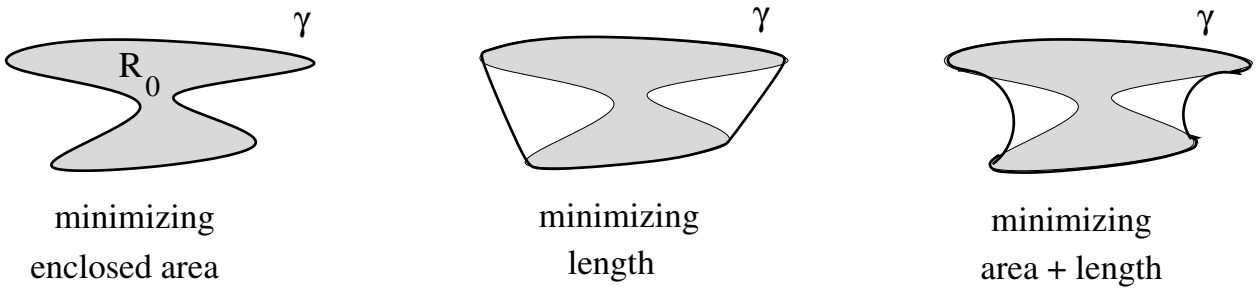


An **admissible strategy** is described by a **set-valued function** $t \mapsto \gamma(t) \subset \mathbb{R}^2$

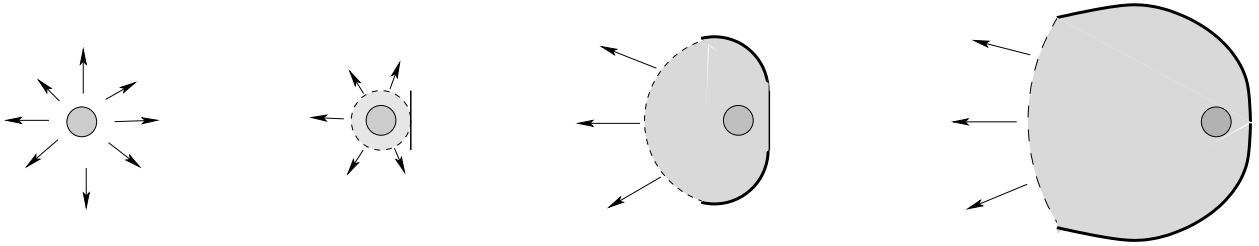
$$\gamma(t) = \text{portion of the wall constructed within time } t$$

Static vs. Dynamic Blocking Problems

Static problems



Dynamic problem



Optimal Confinement Strategies

A **cost functional** should take into account

- The value of the region destroyed by the fire.

- The cost of building the wall.

$\alpha(x)$ = value of a unit area of land around the point x

$\beta(x)$ = cost of building a unit length of wall near the point x

COST FUNCTIONAL

$$J(\gamma) \doteq \lim_{t\to\infty} \left\{ \int_{R^\gamma(t)} \alpha \, dm_2 + \int_{\gamma(t)} \beta \, dm_1 \right\}$$

Mathematical Problems

1. Blocking Problem.

Given an initial set R_0, a multifunction F and a wall construction speed σ, does there exist an admissible strategy $t \mapsto \gamma(t)$ such that **the reachable sets $R^\gamma(t)$ remain uniformly bounded** for all $t > 0$?

2. Optimization Problem.

Find an admissible strategy $\gamma \in \mathcal{S}$ which minimizes the cost functional $J(\gamma)$.

(i) **Existence** of an optimal solution

(ii) **Necessary conditions** for the optimality of an admissible strategy $\gamma(\cdot)$

(iii) **Regularity** of the curves $\gamma(t)$ constructed by an optimal strategy

(iv) **Sufficient conditions** for the optimality of a strategy $\gamma(\cdot)$

(v) **Numerical computation** of an optimal strategy

Equivalent Formulation

(A.B. - T. Wang, *Control Optim. Calc. Var.* 2009)

Blocking Problems and Optimization Problems can be reformulated in terms of one single rectifiable set Γ

$$\text{(strategy)} \qquad t \mapsto \gamma(t) \qquad \longleftrightarrow \qquad \Gamma \qquad \text{(single wall)}$$

$$\gamma(\cdot) \qquad \longrightarrow \qquad \Gamma \doteq \left(\bigcup_{t>0} \gamma(t) \right) \setminus \left(\bigcup_{t>0} R^{\gamma}(t) \right) \qquad \text{(useful walls)}$$

$$\Gamma \qquad \longrightarrow \qquad \gamma(t) \doteq \Gamma \cap \overline{R^{\Gamma}(t)} \qquad \text{(walls touched by the fire within time } t\text{)}$$

Complete strategies

A rectifiable set Γ is **complete** if it contains all of its points of positive upper density:

$$\theta^*(\Gamma; x) \;\doteq\; \limsup_{r\to 0+} \frac{m_1\Big(B(x,r)\cap\Gamma\Big)}{r} \;>\; 0 \qquad \Longrightarrow \qquad x\in\Gamma\,.$$

The **reachable sets** for the differential inclusion

$$\dot{x} \;\in\; F(x) \qquad\qquad x(0)\in R_0 \qquad\qquad (DI)$$

without crossing Γ are

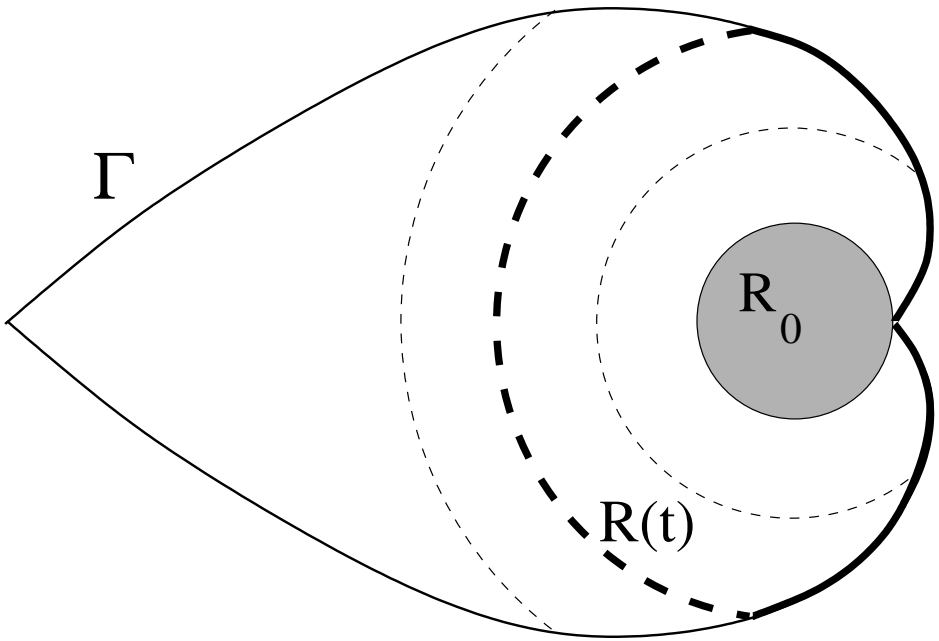
$$R^\Gamma(t) \doteq \Bigg\{ x(t)\,;\;\; x(\cdot)\;\text{ absolutely continuous}\,,\;\; x(0)\in R_0$$
$$\dot{x}(\tau)\in F\big(x(\tau)\big)\quad \text{for a.e. } \tau, \qquad x(\tau)\notin\Gamma\;\text{ for all } \tau\in[0,t]\Bigg\}$$

Admissible curves

The complete, rectifiable set Γ is **admissible** for the differential inclusion (DI) and the construction speed σ if, for every $t \geq 0$, the set

$$\gamma(t) \doteq \Gamma \cap \overline{R^{\Gamma}(t)}$$

has total length $\quad m_1(\gamma(t)) \;\leq\; \sigma t$



$\gamma(t) =$ portion of Γ that needs to be in place by time t

Equivalent Formulations

$$R^\Gamma_\infty \doteq \bigcup_{t\geq 0} R^\Gamma(t) \;=\; \text{[total region burned by the fire]}$$

Blocking Problem 2: Find an admissible rectifiable set $\Gamma \subset \mathbb{R}^2$ such that the corresponding reachable set R^Γ_∞ is bounded.

Optimization Problem 2: Find an admissible rectifiable set $\Gamma \subset \mathbb{R}^2$ which minimizes the cost

$$J(\Gamma) \;=\; \int_{R^\Gamma_\infty} \alpha \, dm_2 + \int_\Gamma \beta \, dm_1$$

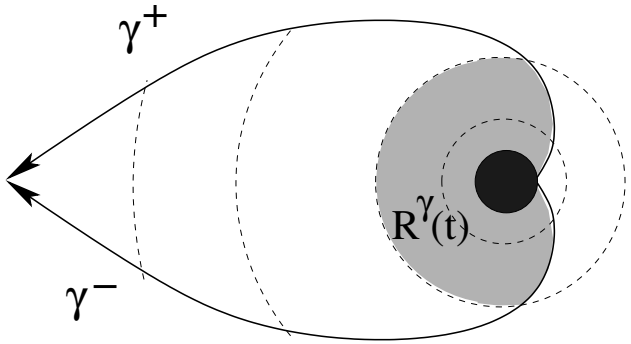
Blocking the Fire

- Fire propagates in all directions with unit speed: $F(x) = B_1$
- Wall is constructed at speed σ

Theorem *(A.B., J.Differential Equations, 2007)*

On the entire plane, the fire can be blocked if $\sigma > 2$, it cannot be blocked if $\sigma < 1$.

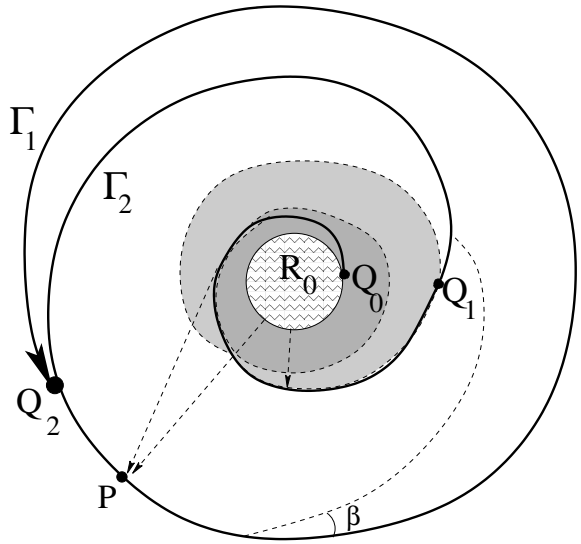
Blocking Strategy: If $\sigma > 2$, construct two arcs of logarithmic spirals along the edge of the fire

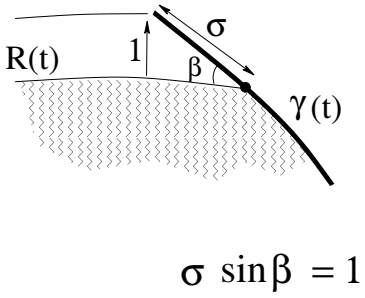


$$\gamma(t) \doteq \left\{ (r,\theta)\,; \quad r = e^{\lambda|\theta|}\,, \quad 1 \le r \le 1+t \right\}, \qquad \lambda \doteq \frac{1}{\sqrt{\frac{\sigma^2}{4} - 1}}$$

When can the fire be blocked ?

Conjecture: Assume the fire propagates with speed 1 in all directions.
On the entire plane the fire can be blocked if and only if $\sigma > 2$



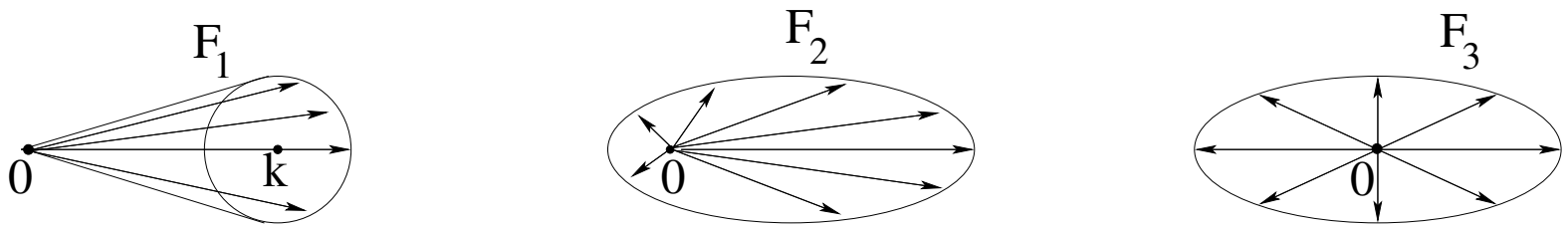


Single spiral strategy: curve closes on itself if and only if
$\sigma > \sigma^{\dagger} = 2.614430844\ldots$ (M. Burago, 2006)

Non-isotropic fire propagation

$$\text{Assume :} \qquad F = \Big\{ (r\cos\theta\,,\ r\sin\theta)\,; \quad 0 \le r \le \rho(\theta) \Big\}$$

$$\rho(-\theta) = \rho(\theta)\,, \qquad 0 \le \rho(\theta') \le \rho(\theta) \qquad \text{for all } \ 0 \le \theta \le \theta' \le \pi\,.$$



Theorem. *(A.B., M. Burago, A. Friend, J. Jou, Analysis and Applications, 2008)*

If the wall construction speed satisfies

$$\sigma \ > \ [\text{vertical width of } F] \ = \ 2 \max_{\theta \in [0,\pi]} \ \rho(\theta)\sin\theta$$

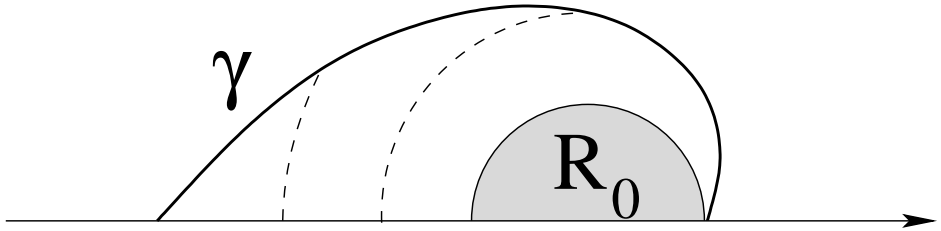
then, for every bounded initial set R_0, a blocking strategy exists

The isotropic case on the half plane

- Fire propagates in all directions with unit speed. $F(x) = B_1$
- Wall is constructed at speed σ

Theorem. (A.B. - T.Wang, *J.Math Anal.Appl.* 2009)

Restricted to a half plane, the fire can be blocked if and only if $\sigma > 1$

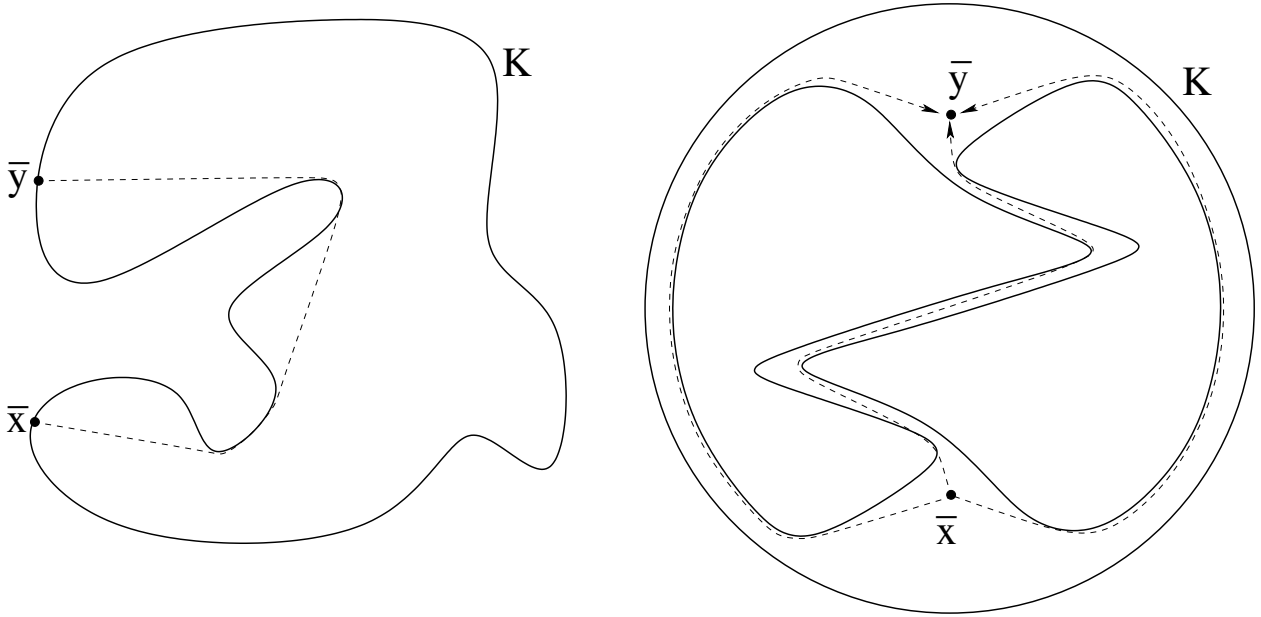


BLOCKING STRATEGY: If $\sigma > 1$, the fire can be enclosed between the horizontal axis and an arc of logarithmic spiral

$$\lambda \doteq \frac{1}{\sqrt{\sigma^2 - 1}} \qquad \Gamma \doteq \left\{ (r, \theta); \quad r = e^{\lambda \theta}, \quad \theta \in [0, \pi] \right\}$$

Regularity of the distance function

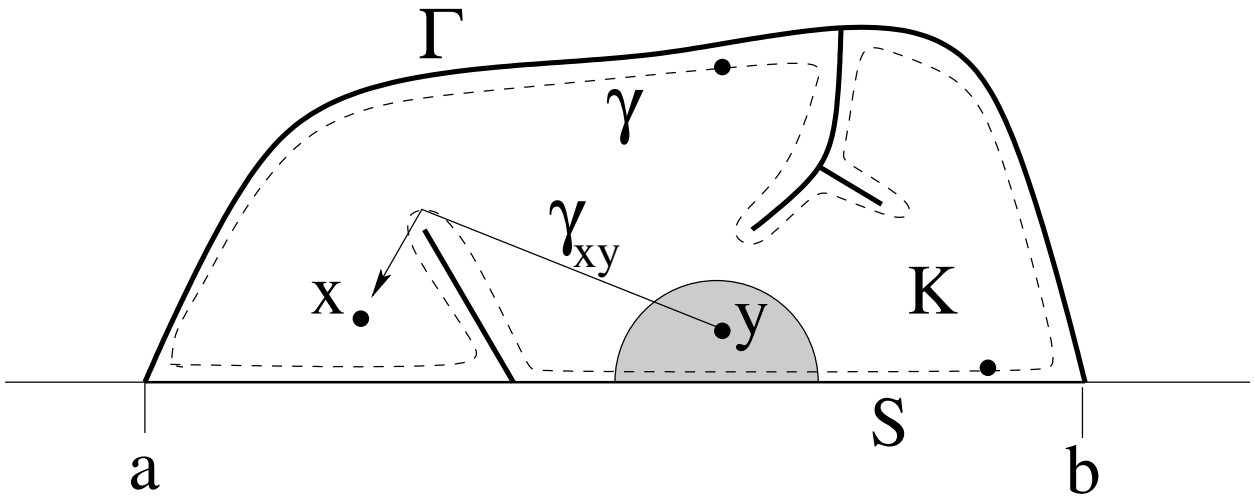
$d_K(x,y) \doteq$ minimum length among all paths $\gamma : [0,1] \mapsto K$ joining x with y



Lemma. If $K \subset \mathbb{R}^2$ is compact, simply connected, then the map $y \mapsto d_K(x,y)$ is $\mathcal{C}^1$ in the interior of $K \setminus \{x\}$.

$\sup_{x,y \in K} \; d_K(x,y)$ is attained at boundary points.

Estimates on the distance function



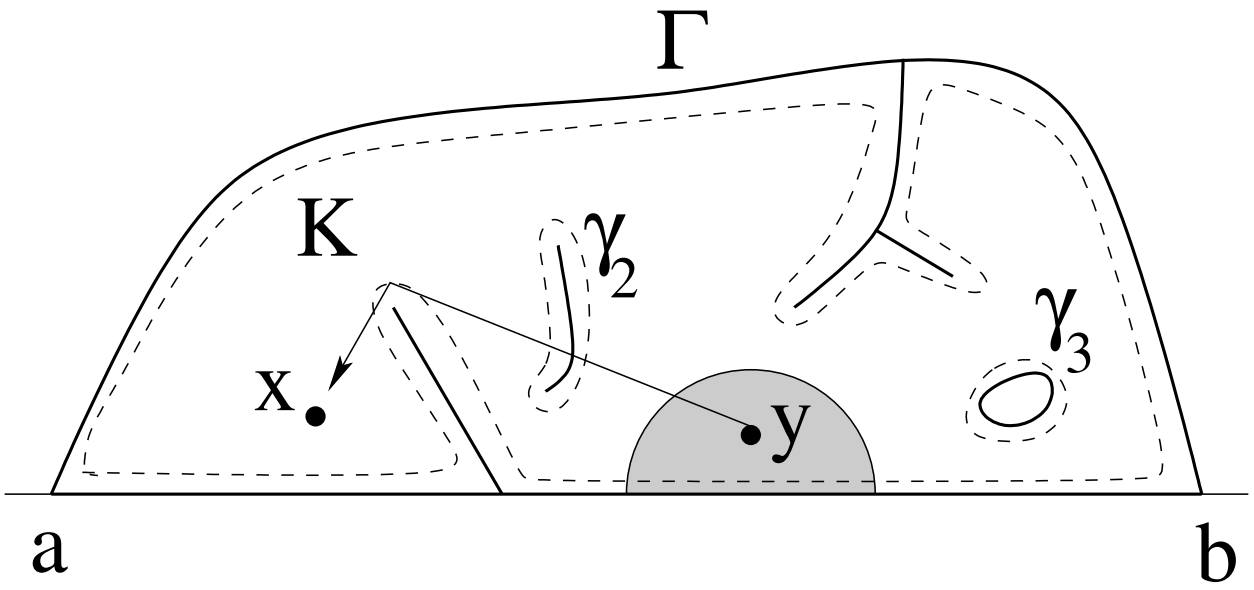
Assume: barrier Γ is completed at time $T > 0$

S = segment joining the points a, b

Any two points $x, y \in K$ can be connected by a path γ_{xy} of length

$$m_1(\gamma_{xy}) \;\leq\; \frac{1}{2} m_1\Big(\Gamma \cup S\Big) \;\leq\; \frac{1}{2}\Big(\sigma T + (b-a)\Big) \;<\; \sigma T$$

Same conclusion in the case of several connected components



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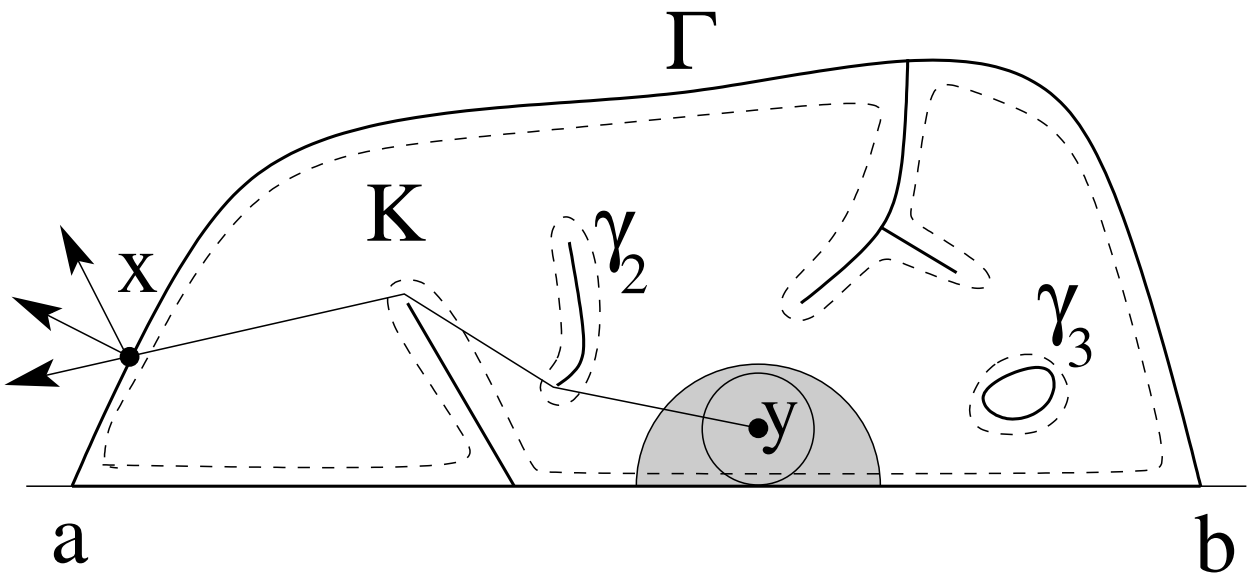
$$m_1(\gamma_{xy}) \;\leq\; \frac{1}{2} m_1\Big(\Gamma \cup S\Big) \;\leq\; \frac{1}{2}\Big(\sigma T + (b-a)\Big) \;<\; \sigma T$$

No strategy can block the fire if $\sigma \leq 1$



$x =$ position of "last brick of the wall"

Fire reaches x and spreads outside, before time T when the barrier is completed

$$d_K(x,y) \;\leq\; \max_{x \in \partial K} d_K(x,y) \;\leq\; \frac{1}{2} m_1(\Gamma \cup S) \;\leq\; m_1(\Gamma) \;\leq\; \sigma T$$

Existence of Optimal Strategies

Fire propagation: $\dot{x} \in F(x)$ $\qquad x(0) \in R_0$

Wall constraint: $\int_{\gamma(t)} \psi \, dm_1 \leq t$ $\qquad (1/\psi(x) = \text{construction speed at } x)$

Minimize: $J(\gamma) = \left\{ \int_{R^\gamma(t)} \alpha \, dm_2 + \int_{\gamma(t)} \beta \, dm_1 \right\}$

Assumptions:

(A1) The initial set R_0 is open and bounded. Its boundary satisfies $m_2(\partial R_0) = 0$.

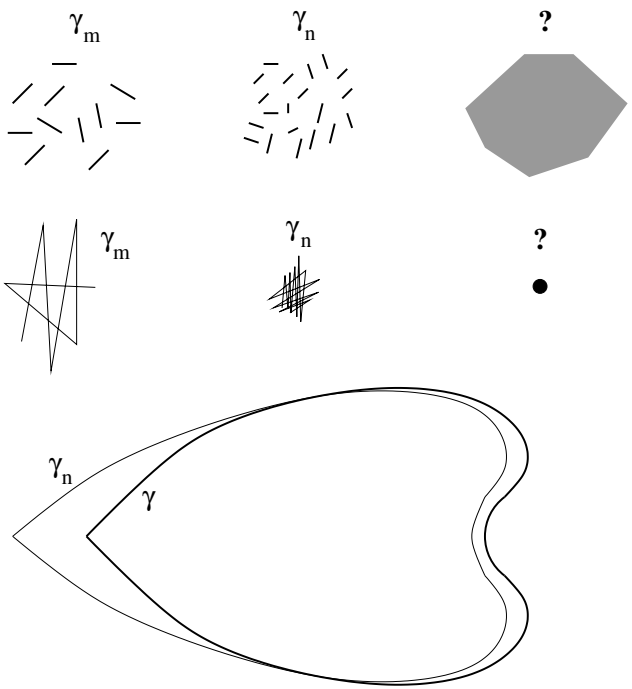
(A2) The multifunction F is Lipschitz continuous w.r.t. the Hausdorff distance. For each $x \in \mathbb{R}^2$ the set $F(x)$ is compact, convex, and contains a ball of radius $\rho_0 > 0$ centered at the origin.

(A3) For every $x \in \mathbb{R}^2$ one has $\alpha(x) \geq 0$, $\beta(x) \geq 0$, and $\psi(x) \geq \psi_0 > 0$. α is locally integrable, while β and ψ are both lower semicontinuous.

Theorem (A.B. - C. De Lellis, *Comm. Pure Appl. Math.* 2008)

Assume (A1)-(A3), and $\inf_{\gamma \in \mathcal{S}} J(\gamma) < \infty$.
Then the minimization problem admits an optimal solution γ^*.

Direct method: Consider a minimizing sequence of strategies $\gamma_n(\cdot)$
Define the optimal strategy γ^* as a suitable limit.



STEP 1: Replace the $\gamma_n(\cdot)$ by **complete strategies** γ_n^c

$$\theta^*(E,x) \doteq \limsup_{r\downarrow 0} \frac{m_1(B(x,r)\cap E)}{2r} \qquad \text{(upper density)}$$

$$\overline{\gamma}_n(t) \doteq \gamma_n(t) \cup \left\{x\in\mathbb{R}^2;\ \theta^*(\gamma_n(t),x)>0\right\}, \qquad \gamma_n^c(t) \doteq \bigcap_{s>t}\overline{\gamma}_n(s)$$

STEP 2: For each rational time τ, order the connected components of $\gamma_n(\tau)$ according to decreasing length:

$$\ell_{n,1}\geq \ell_{n,2}\geq \ell_{n,3}\geq\cdots \qquad \gamma_n(t)=\gamma_{n,1}\cup\gamma_{n,2}\cup\gamma_{n,3}\cup\cdots$$

Taking a subsequence, as $n\to\infty$ we can assume

$$\ell_{n,i}(\tau)\to\ell_i(\tau) \qquad \gamma_{n,i}(\tau)\to\gamma_i(\tau) \qquad \tau\in\mathbf{Q}$$

We then define $\gamma(\tau)\doteq \bigcup_{i\geq 1,\,\ell_i(\tau)>0}\gamma_i(\tau)$. Finally $\gamma^*(\cdot)\doteq$ completion of $\gamma(\cdot)$.

STEP 3: Lower semicontinuity of ψ, β imply

$$\int_{\gamma^*(t)} \psi \, dm_1 \;\leq\; \liminf_{n\to\infty} \int_{\gamma_n(t)} \psi \, dm_1 \;\leq\; t \qquad\qquad \int_{\gamma^*(t)} \beta \, dm_1 \leq \liminf_{n\to\infty} \int_{\gamma_n(t)} \beta \, dm_1$$

Hence the limit strategy $\gamma(\cdot)$ is admissible.

STEP 4:

$$\int_{R^{\gamma^*}(t)} \alpha \, dm_2 \;\leq\; \liminf_{n\to\infty} \int_{R^{\gamma_n}(t)} \alpha \, dm_2$$

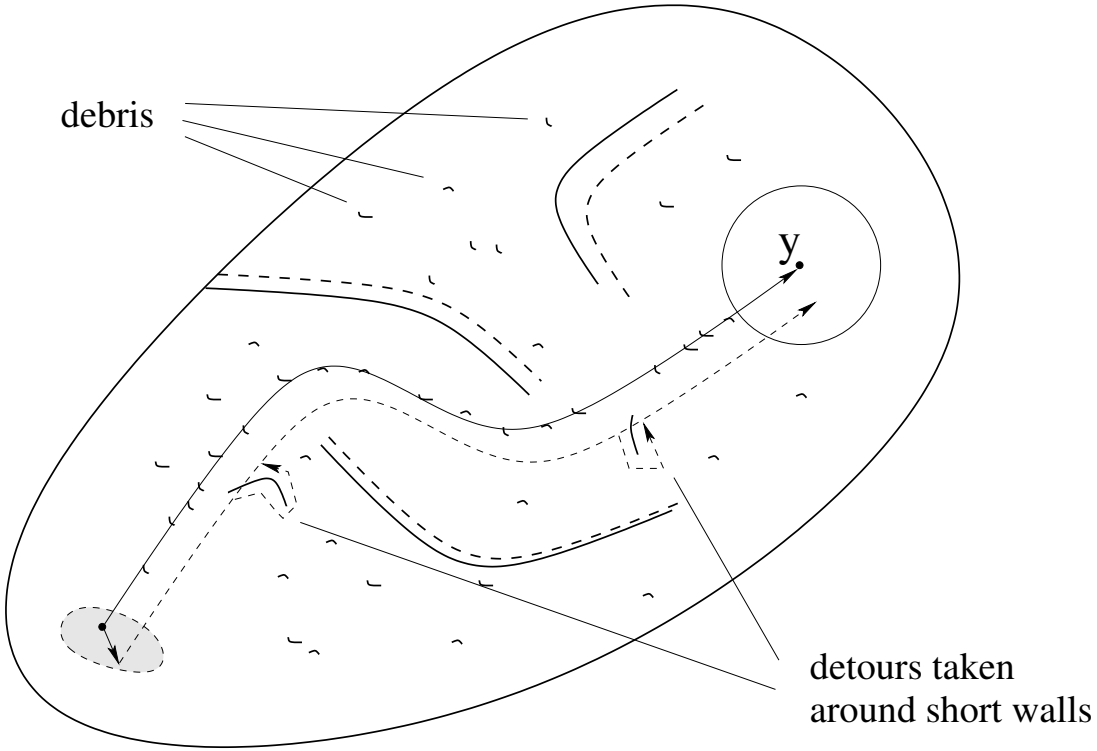
Claim: given $y \in R^{\gamma^*}(t)$ and $r > 0$, for every n large: $R^{\gamma_n}(t) \cap B(y, r) \neq \emptyset$

Given $\varepsilon > 0$, decompose: $$\gamma_n(\tau) \;=\; \left(\bigcup_{i \leq i_1(\tau)} \gamma^{\tau}_{n,i} \right) \cup \gamma'_n(\tau) \cup \gamma''_n(\tau)$$

[large connected components] $\cup$ [small connected components] $\cup$ [debris]

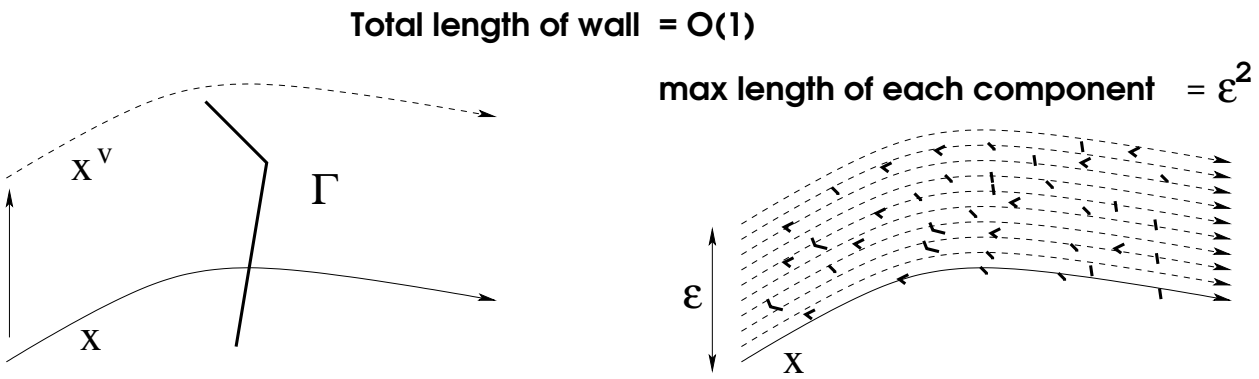
$$m_1(\gamma'_n(\tau)) < \varepsilon \qquad\qquad m_1(\gamma''_n(\tau)) = \mathcal{O}(1)$$

$$\lim_{n\to\infty} \text{[maximum length of connected components in } \gamma''_n] \;= 0$$



• Shift the trajectory $x^\sharp$ so that it crosses an amount $\mathcal{O}(\varepsilon)$ of debris (connected components of γ_n'') for all n.

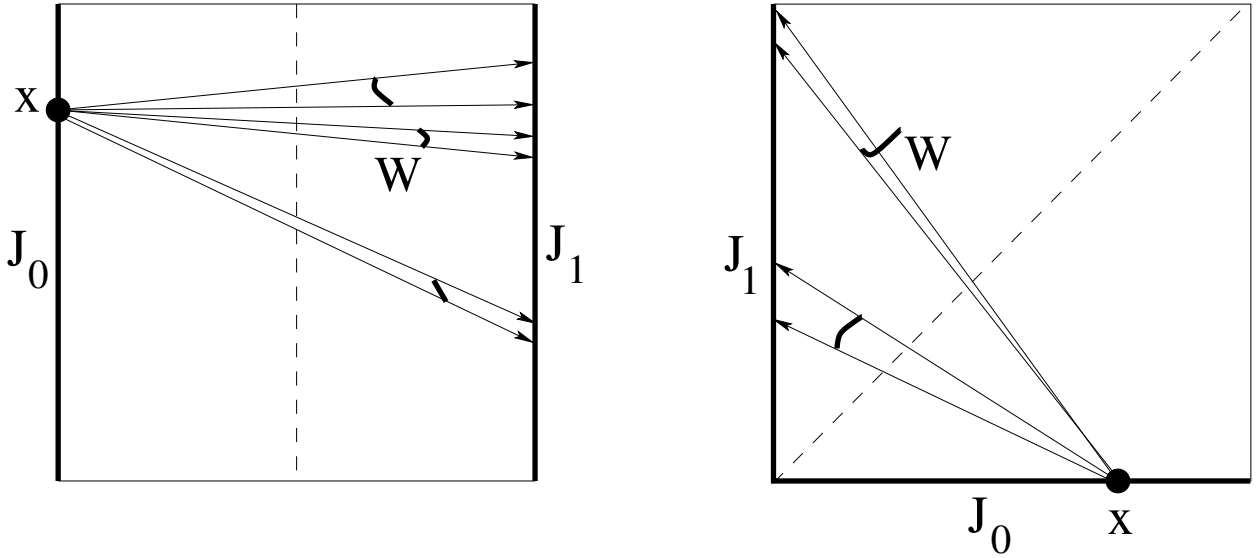
• Take detours of total length $\mathcal{O}(\varepsilon)$ to get around the small connected components γ_n' and the debris.



Given $x(\cdot)$, find a shifted trajectory $x^v(\cdot)$ so that

$$[\text{total length of connected components of } \Gamma \text{ crossed by } x^v(\cdot)] \; < \; \varepsilon$$

Lemma 1: In case 2, this can be achieved by a shift $|v| = \mathcal{O}(\varepsilon)$

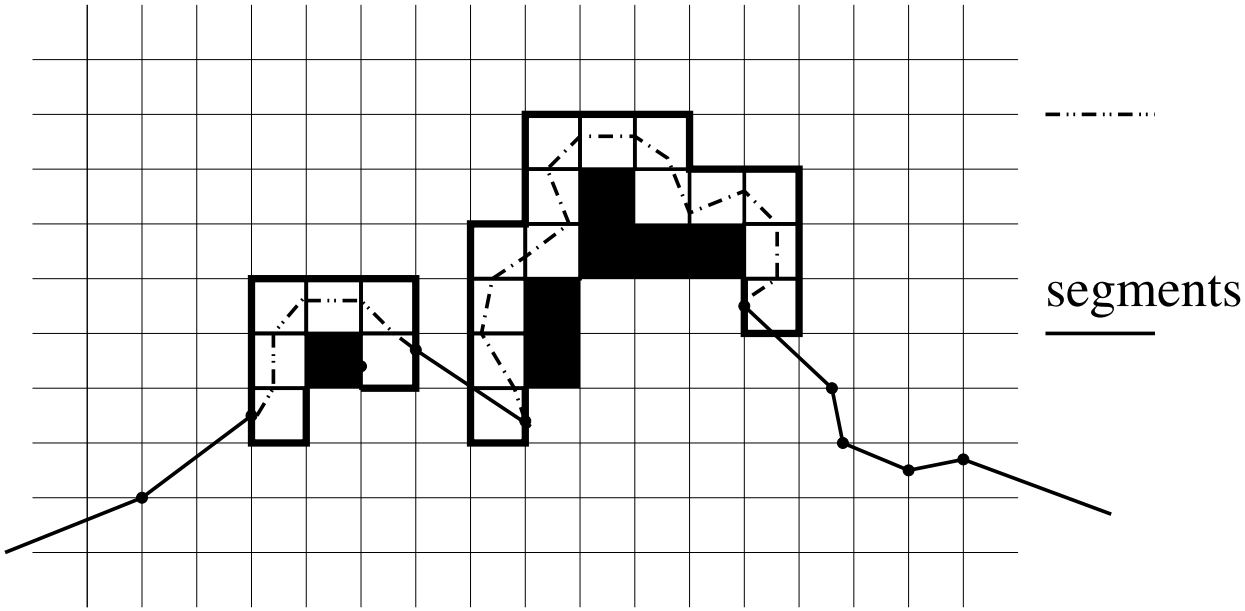


Lemma 2: J_0, J_1 edges of the unit square Q. For any $\mu > 0$ there exists $\kappa > 0$ such that:

If $W \subset Q$ is a set with $m_1(W) < \kappa$, then there is a set $K_0 \subseteq J_0$

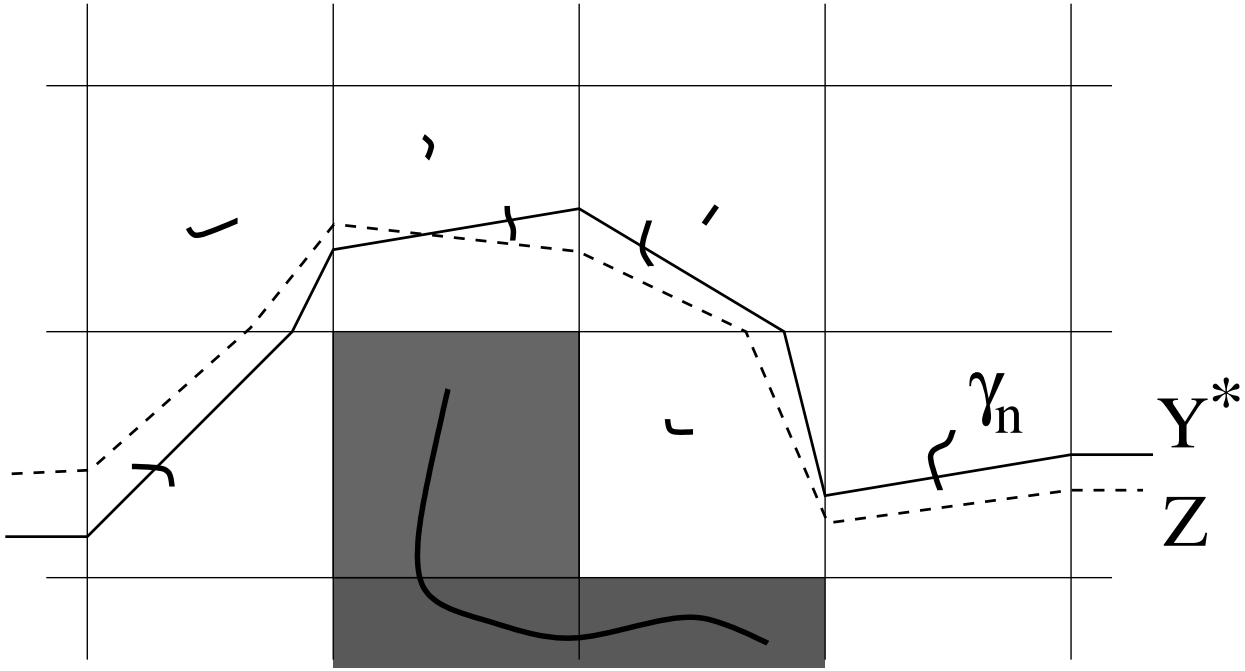
(i) $m_1(K_0) \geq 1 - \mu$,

(ii) For every $x \in K_0$, the set J_x of points $y \in J_1$ for which the segment $[x, y]$ with endpoints x, y does not intersect W has measure $m_1(J_x) \geq 1 - \mu$.



- Construct a σ-grid
- Replace trajectory by a polygonal Y^σ
- If $m_1(\gamma_n \cup Q) > \kappa\sigma$ the square Q is black. Otherwise it is white.
- Construct polygonal detours around the black islands, with total length $\mathcal{O}(\varepsilon)$

- Modify the trajectory Y^* inside white squares
  so that the new polygonal Z does not cross any wall of γ_n



Alternative approach: the minimum time function

(C.DeLellis & R.Robyr, 2010)

$$\dot{x} \in F(x) \qquad\qquad x(0) \in R_0 \subset \mathbb{R}^2$$

Assume: $B(0,\rho) \subseteq F(x)$

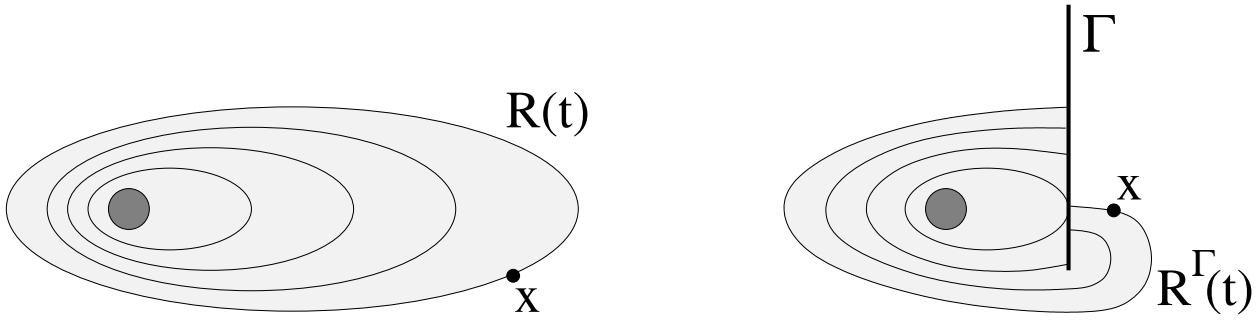
Minimum time function: $\quad T(x) \doteq \inf\{t \geq 0\,;\ \ x \in R(t)\}$

is Lipschitz continuous and satisfies the Hamilton-Jacobi equation

$$H(x, \nabla T(x)) = 0 \qquad\qquad a.e.$$

$$H(x,p) = \max_{v \in F(x)} \ \langle v\,,\ p\rangle - 1$$

The minimum time function with obstacle



Barrier: a complete, rectifiable set $\Gamma \subset \mathbb{R}^2$

Minimum time function with obstacle: $\quad T^\Gamma(x) \doteq \inf\{t \geq 0\,;\ \ x \in R^\Gamma(t)\}$

$R^\Gamma(t)$ = set of points reached by F-trajectories which start in R_0 and do not cross Γ

Goal: characterize T^Γ as the unique maximal subsolution to a H-J equation

A family $\mathcal{S}^\Gamma$ of subsolutions

Definition. A function $u : \mathbb{R}^2 \mapsto [0, \infty]$ is in the set $\mathcal{S}^\Gamma$ if

- $u \in SBV \qquad (Du = \nabla u + D^{jump} u + D^{Cantor} u, \quad \text{with} \quad D^{Cantor} u \equiv 0)$
- $m_1(J_u \setminus \Gamma) = 0 \qquad (J_u \doteq \text{set of jump points of } u)$
- $u = 0 \;\; \text{on} \;\; R_0$
- $H(x, \nabla u(x)) \leq 0 \quad \text{for a.e. } x$

$\nabla u \doteq$ absolutely continuous part of the distributional derivative Du.

Theorem. (C.DeLellis & R.Robyr, 2010 - Archive Rat. Mech. Anal.)

Let Γ be a complete, rectifiable set. Then the minimum time function T^Γ is the unique maximal element of $\mathcal{S}^\Gamma$.

Corollary. (C.DeLellis & R.Robyr, 2010)

Consider the cost functions $\alpha \geq 0, \beta \geq 0$, with α locally integrable and β lower semicontinuous. Then there exist a blocking strategy Γ which minimizes the cost among all admissible ones.

Proof of the Corollary. Take a minimizing sequence of admissible barriers Γ_k.

Consider the corresponding minimum time functions $T_k \doteq T^{\Gamma_k}$

Using the Ambrosio-De Giorgi compactness theorem for SBV functions, one obtains a convergent subsequence $T_k \to U$, such that

- $U \in SBV$
- the jump set J_U is a rectifiable set, and

$$\int_{J_U} \psi \, dm_1 \;\leq\; \liminf_{k\to\infty} \int_{J_{T_k}} \psi \, dm_1 \;\leq\; t\,.$$

Hence $\Gamma \doteq [\text{completion of } J_U]$ is an admissible barrier.

$$\int_{R^\Gamma_\infty} \alpha \, dm_2 \;\leq\; \liminf_{k\to\infty} \int_{R^{\Gamma_k}_\infty} \alpha \, dm_2 \qquad\qquad \int_{\Gamma} \alpha \, dm_2 \;\leq\; \liminf_{k\to\infty} \int_{\Gamma_k} \alpha \, dm_2$$

Necessary conditions for optimality

Problem: find an admissible barrier Γ which minimizes

$$J(\Gamma) \doteq \alpha \cdot [\text{total burned area}] + \beta \cdot [\text{length of the curve}]$$

GOAL: derive a set of ODE's describing the walls built by an optimal strategy

- A.B., *J.Differential Equations*, 2007
- A.B. - T.Wang, *ESAIM, Control Optim. Calc. Var.* 2010
- T.Wang, *Intern. J. Control*, to appear.

Classification of arcs in an optimal strategy

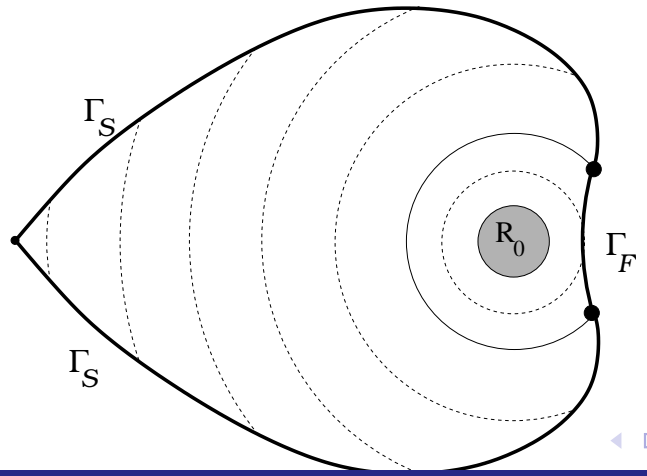
Minimum time function $\quad T^{\Gamma}(x) \doteq \inf \left\{ t \geq 0 ; \quad x \in \overline{R^{\Gamma}(t)} \right\}$

Set of times where the constraint is *saturated*

$$\mathcal{S} \doteq \left\{ t \geq 0 ; \quad meas \left(\Gamma \cap \overline{R^{\Gamma}(t)} \right) = \sigma t \right\}$$

Boundary arcs: $\quad \Gamma_{\mathcal{S}} \doteq \{ x \in \Gamma ; \quad T^{\Gamma}(x) \in \mathcal{S} \}$
constructed along the advancing fire front

Free arcs: $\quad \Gamma_{\mathcal{F}} \doteq \{ x \in \Gamma ; \quad T^{\Gamma}(x) \notin \mathcal{S} \}$
constructed away from the fire front

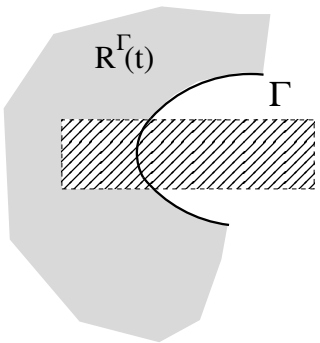


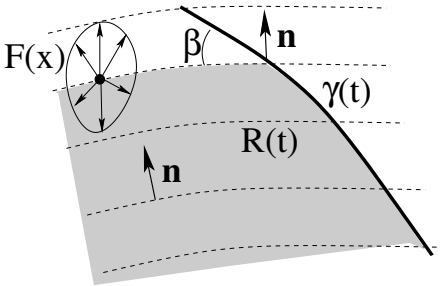
Optimality conditions, minimizing the value of burned area

1. A free arc Γ. The curvature must be proportional to the local value of the land

$$r(s) = \text{radius of curvature} \qquad \alpha = \text{land value}$$

$$r(s) \cdot \alpha\big(\Gamma(s)\big) = \text{const.}$$

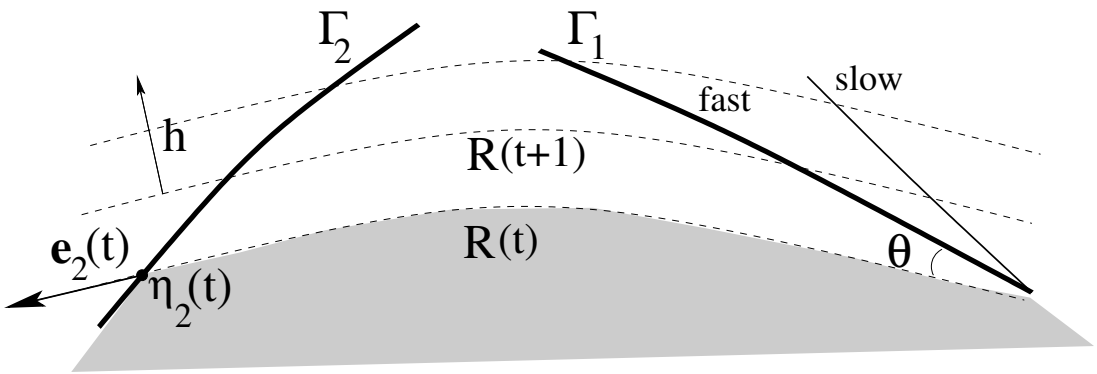




2. A single boundary arc Γ. The wall is constructed at maximum speed σ, always remaining at the edge of the burned set

$$\sigma \sin\beta \;=\; \max_{y \in F(x)} \mathbf{n} \cdot y$$

3. Two or more boundary arcs: $\Gamma_1, \ldots, \Gamma_\nu$, constructed simultaneously for $t \in [a, b]$



Sum of construction speeds $\leq \sigma$

At which speed should each wall be constructed ?

(Solution is found applying Pontryagin's Maximum Principle)

Value of Time

$$J(\Gamma) \doteq \alpha \cdot [\text{total burned area}] + \beta \cdot [\text{length of the curve}]$$

There exists a non-increasing scalar function $t \mapsto W(t)$

$\approx$ Lagrange multiplier corresponding to the constraint

$$m_1\left(\Gamma \cap \overline{R^\Gamma(t)}\right) \leq \sigma t$$

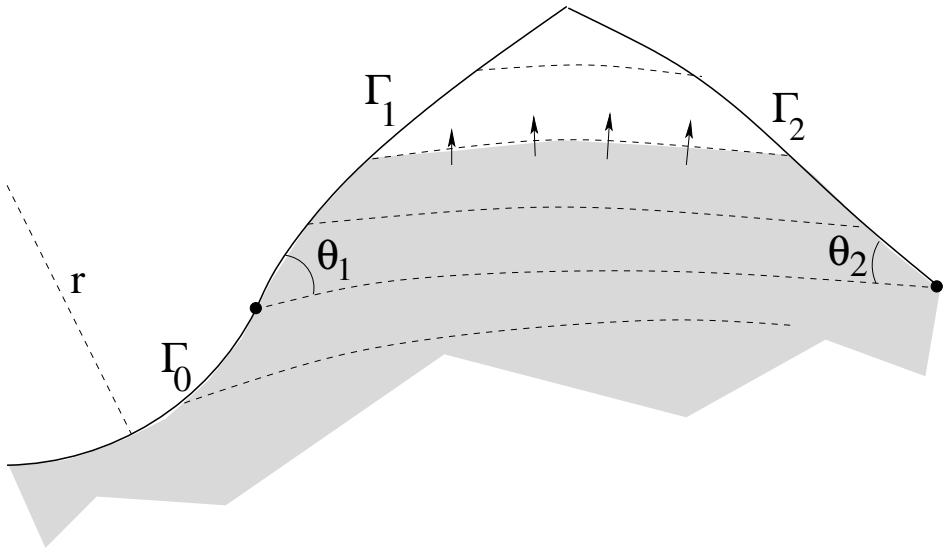
1. A free arc Γ_0: $\qquad W(t) \;=\; (\alpha r - \beta)\sigma \;=\;$ constant

$r =$ radius of curvature

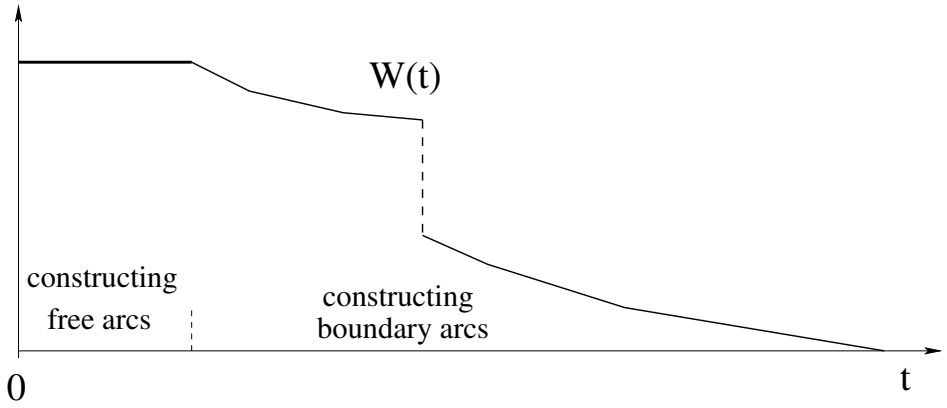
2. Two or more boundary arcs: $\Gamma_1, \ldots, \Gamma_\nu$, constructed simultaneously

$$W(t) \;=\; W_i(t) \;=\; \left(\frac{q_i(t)}{\cos\theta_i(t)} - \beta \right) \sigma$$

$q \;=\; (q_1, q_2, \ldots, q_\nu) \;=\;$ adjoint variable in the PMP.



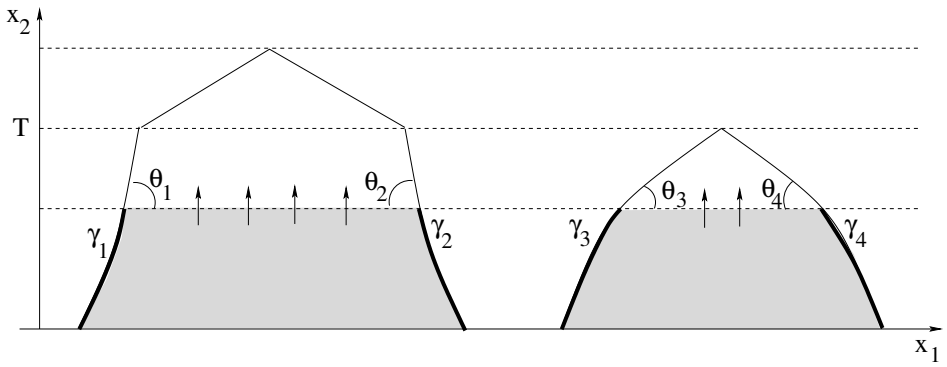
Instantaneous value of time



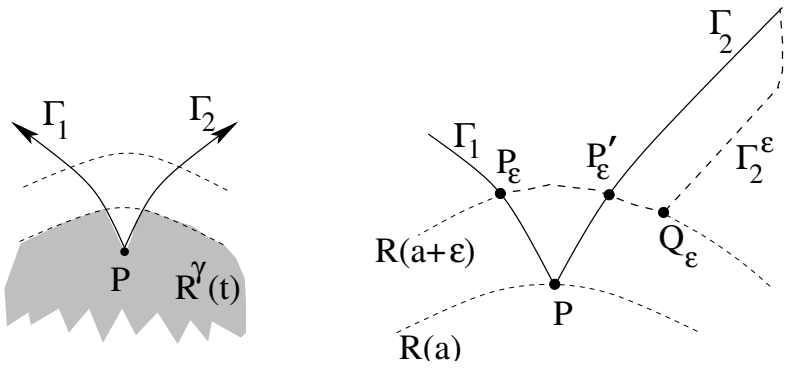
$t \mapsto W(t)$ is non-increasing

- continuous at times where a free arc joins a boundary arc (tangentially)

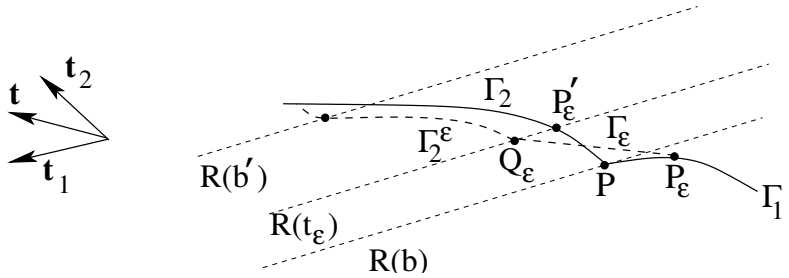
- jumps down at times where two boundary arcs join together



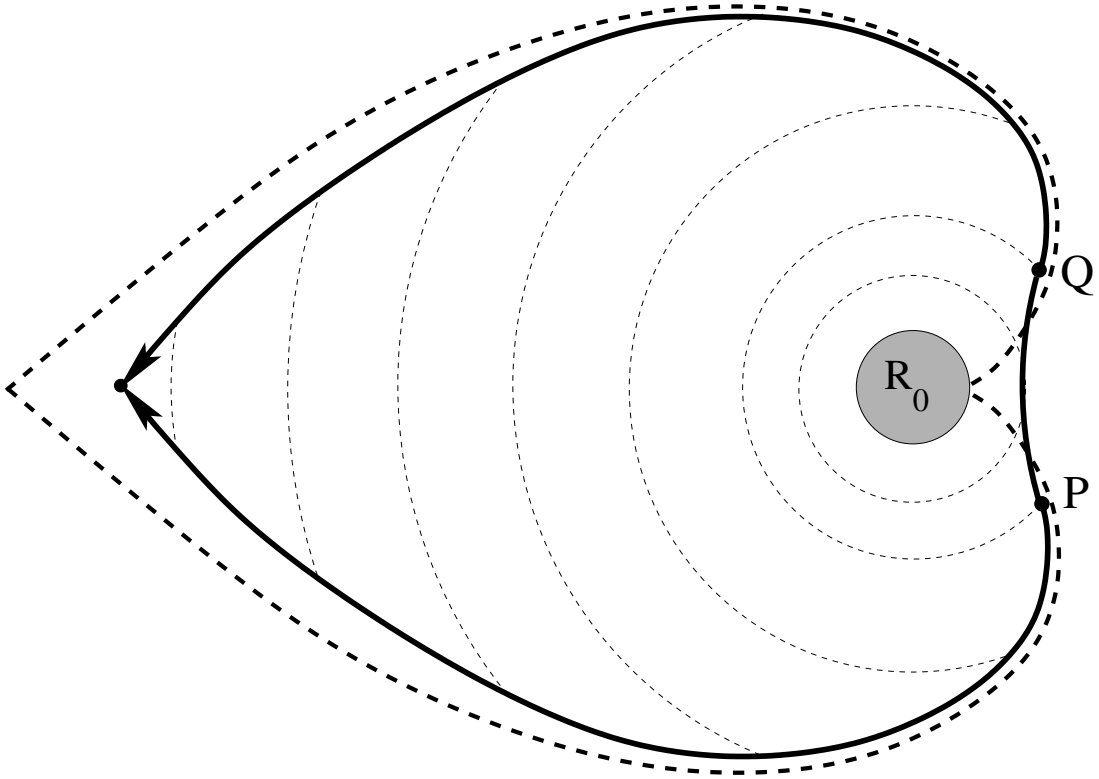
Junctions between different arcs



Two boundary arcs originating at the same point are not optimal



Non-parallel junctions between a free arc and a boundary arc are not optimal

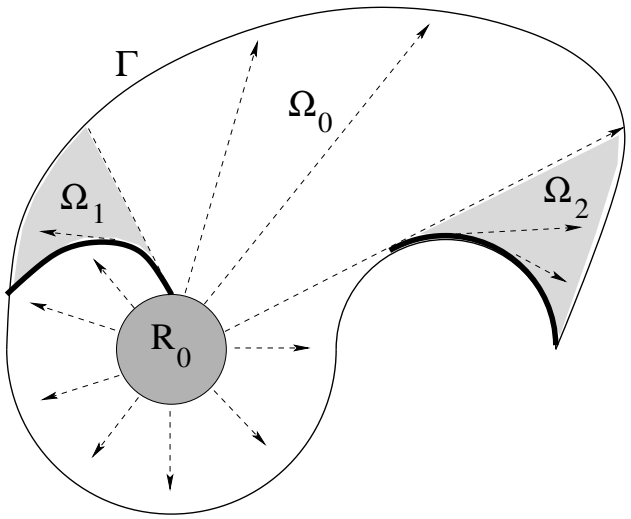


circle + two spirals

(is better than two spirals only)

Further classification

- blocking arcs $\Gamma^b \doteq \Gamma \cap \partial R_\infty^\Gamma$
- delaying arcs Γ^d

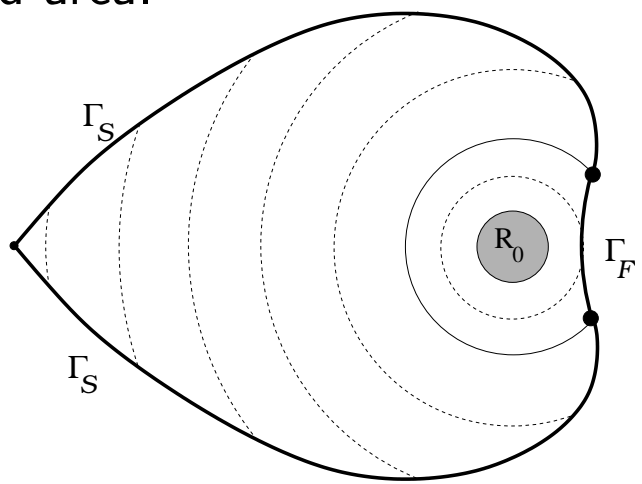


Necessary conditions for the optimality of delaying arcs
(Tao Wang, 2010)

Sufficient conditions for optimality ?

Standard Isotropic Problem:

- Fire starts on the unit disc, propagating with unit speed in all directions.
- Barrier can be constructed at speed $\sigma > 2$.
- Minimize the total burned area.

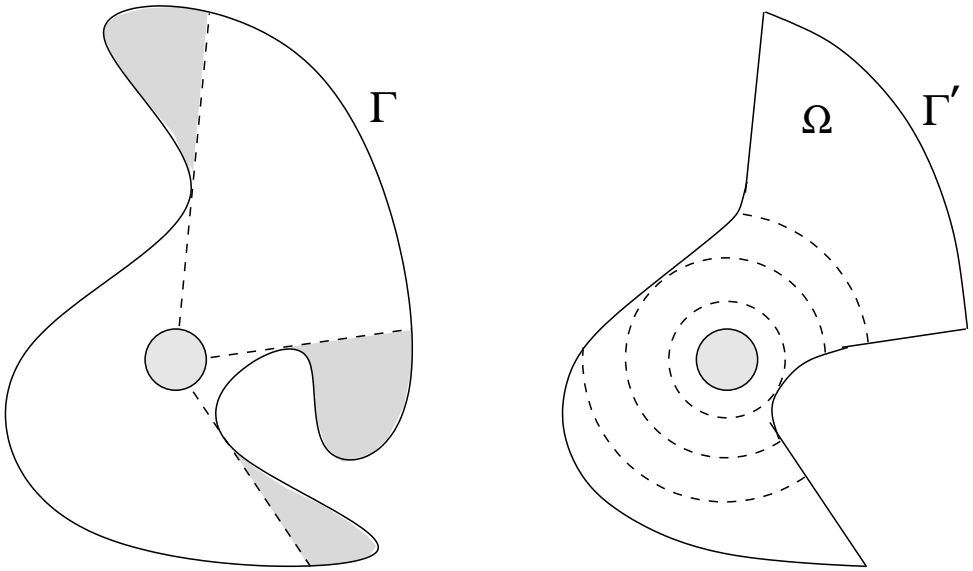


Theorem *(A.B. - T.Wang, 2010)*

The barrier consisting of

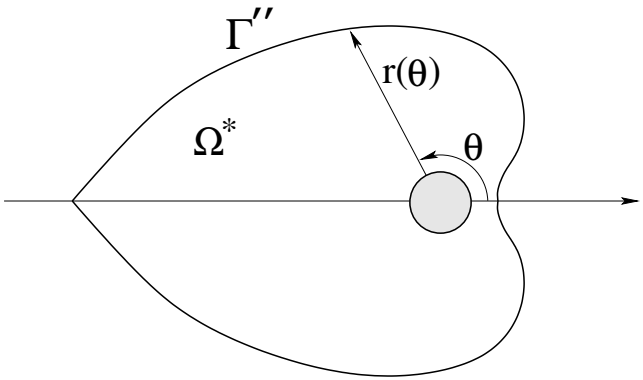
circle + two logarithmic spirals

is optimal among all simple closed curves enclosing R_0



original curve

star shaped



symmetric, nondecreasing rearrangement

Symmetric rearrangement

$$\Omega \;=\; \Big\{(r\cos\theta,\, r\sin\theta)\,;\;\; 0 \;\le\; r \;\le\; \rho(\theta)\Big\}$$

$$\Omega^* \;=\; \Big\{(r\cos\theta,\, r\sin\theta)\,;\;\; 0 \;\le\; r \;\le\; \tilde{\rho}^*(\theta)\Big\},$$

$\rho^* : [-\pi, \pi] \mapsto \mathbb{R}_+$ is the symmetric, nondecreasing rearrangement $r(\cdot)$.

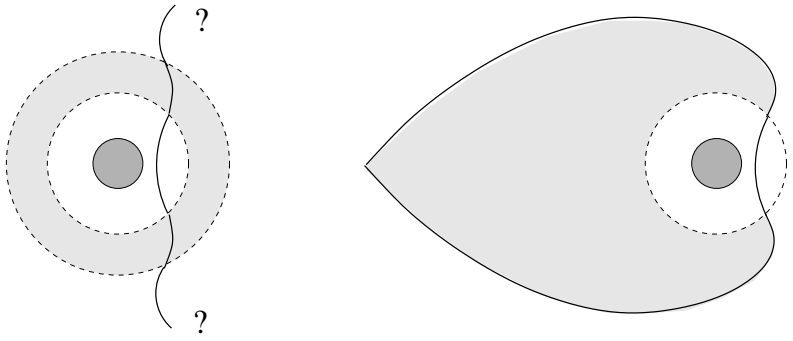
$$\text{meas}\Big(\{\theta\,;\;\; \rho^*(\theta) \;<\; c\}\Big) \;=\; \text{meas}\Big(\{\theta\,;\;\; \rho(\theta) \;<\; c\}\Big) \qquad \text{for all } c > 0$$

Polar coordinate representation: $\theta \mapsto r(\theta)$ non-decreasing, for $\theta \in [0, \pi]$

Admissibility constraint: $$m_1\left(\{x \in \Gamma\,; \quad |x| \leq 1 + t\}\right) \leq \sigma t$$

not saturated $\Longrightarrow$ circumferences

saturated $\Longrightarrow$ logarithmic spirals



Numerical Simulations

(A.B. - T. Wang, *ESAIM; Control Optim. Calc. Var.* 2009)

Assume: only blocking arcs, no delaying arcs.

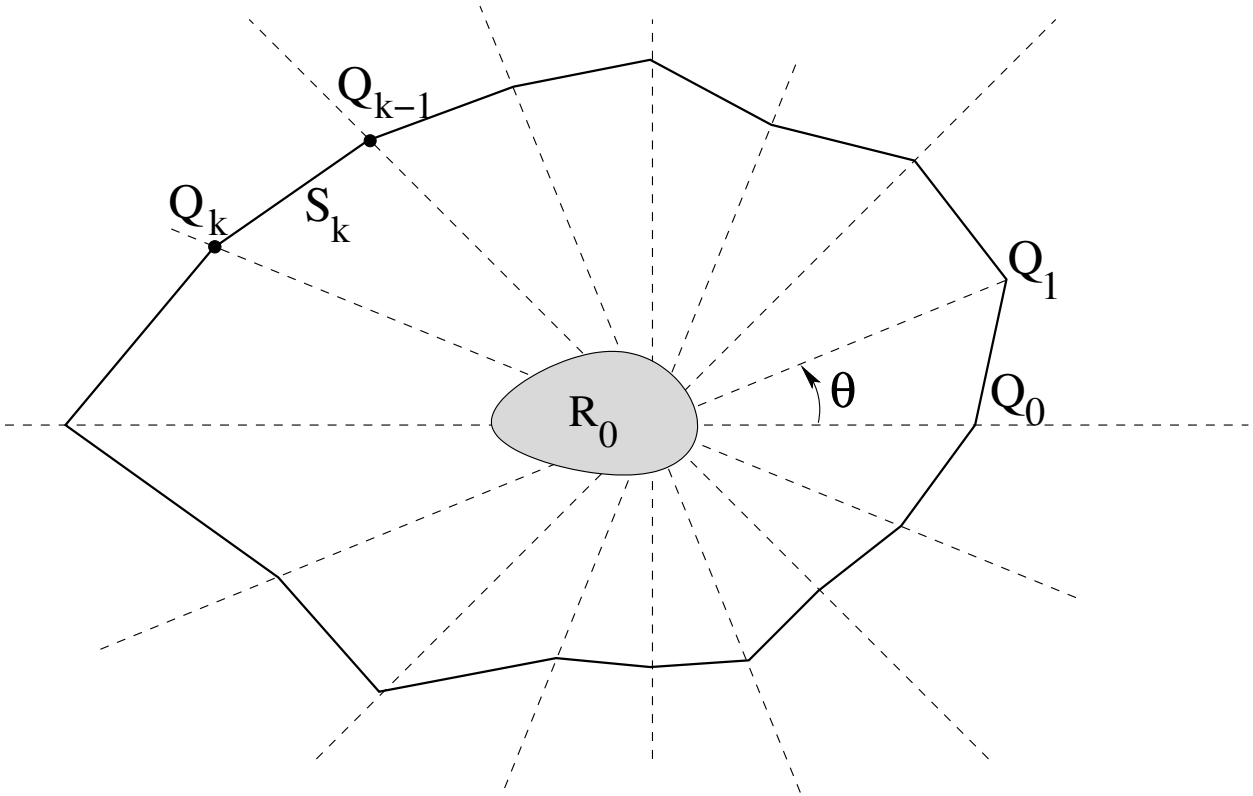
$$\text{minimize total burned area:} \quad m_2\left(R^{\Gamma}_{\infty}\right)$$

subject to

$$m_1\left(\Gamma \cup \overline{R^{\Gamma}(t)}\right) \;\leq\; \sigma t \qquad\qquad \text{for all} \;\; t \geq 0$$

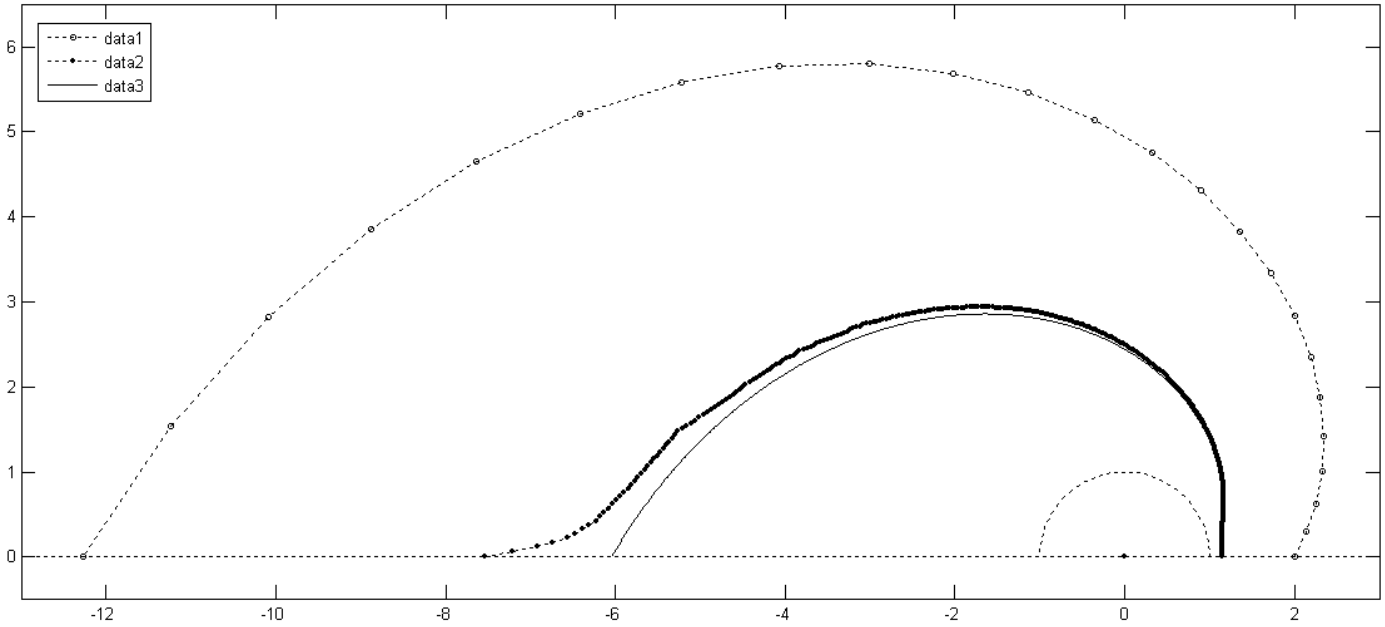
Approximate the barrier with a polygonal:

- fix an angle $\theta = 2\pi/n$
- assign radii $r_k = r(k\theta), \quad k = 1, \ldots, n$
- starting with an admissible polygonal, search for a local minimizer subject to admissibility constraints
- double number of nodes (replace n by $2n$), repeat local minimization ...



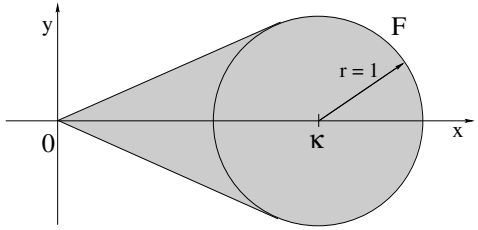
1. The isotropic case

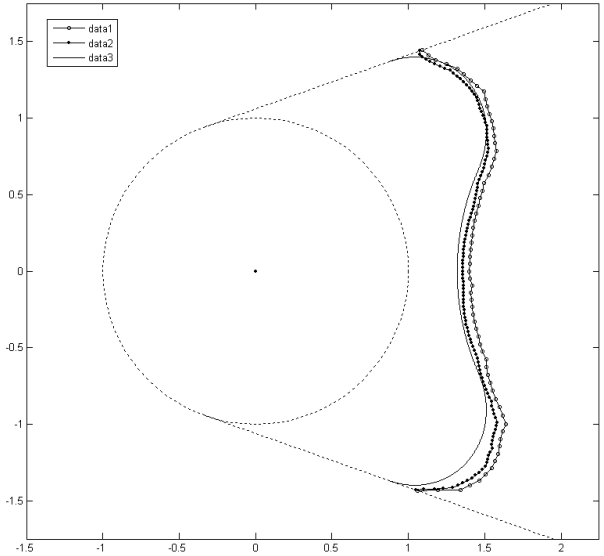
$F(x) = R_0 = B_1$ (unit disc), $\sigma = 4$. Minimize: total burned area.



2. A non-isotropic case

$$F = \left\{ (\lambda x, \lambda y)\,; \quad (x-3)^2 + y^2 \le 1, \quad \lambda \in [0,1] \right\}$$





Choose: $\sigma = 4.1$, $R_0 =$ unit disc.

Analytic solution: A.Friend (2007). Numerical solution: T.Wang (2008)

Some open problems

1 (Isotropic blocking problem). On the whole plane, assume:
• fire propagates with unit speed in all directions.

Conjecture 1: a blocking strategy exists if and only if the wall construction speed is $\sigma > 2$.

2 (Sufficient conditions). Not one single example is known where a blocking strategy can be proved to be optimal.

Conjecture 2: the "circle + two spirals" strategy is optimal for the isotropic problem.

Basic difficulty: delaying arcs

3 (Existence of optimal strategies). Determine whether an optimal strategy exists, in the general case where the velocity sets satisfy $0 \in F(x)$ but without assuming $B(0, \rho) \subset F(x)$
(so that fire propagation speed is not uniformly positive in all directions).

4 (Regularity). If the initial set R_0 has a smooth boundary and the cost functions are smooth, what is the regularity of an optimal strategy ?
Does it produce a finite number of piecewise $\mathcal{C}^1$ arcs ?
Is the optimal barrier connected ?
When is is useful to construct delaying arcs ?

On the minimal speed for a blocking strategy

If Γ is a simple closed curve, then the construction speed must be $\sigma > 2$.

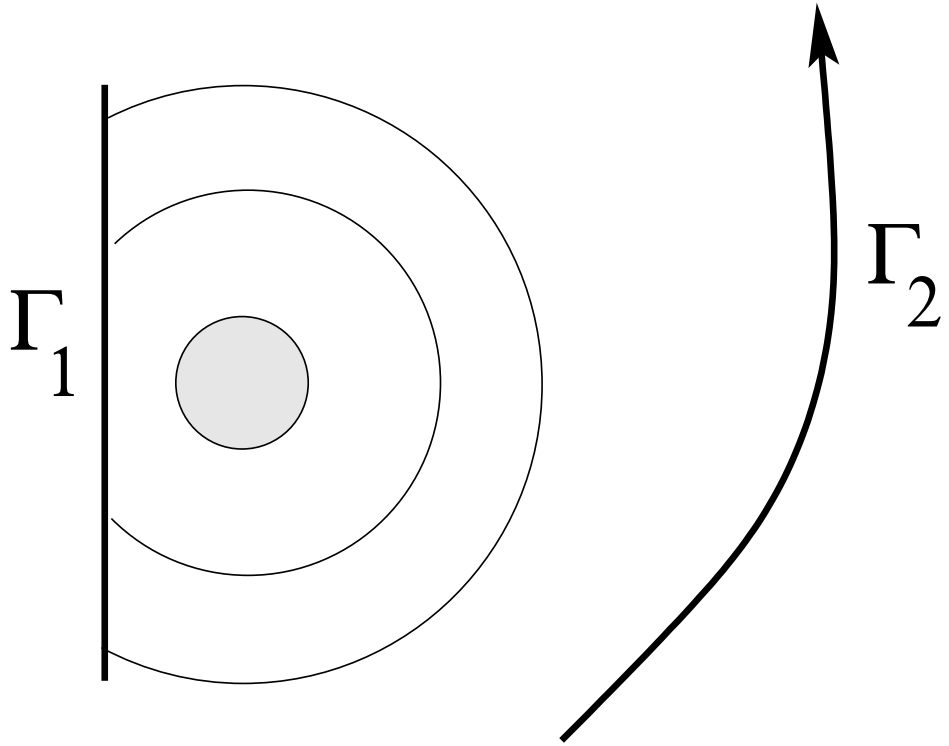
Indeed, let $P=$ last brick of the wall. Then

$$m_1(\omega) \; < \; \frac{1}{2}\, m_1(\Gamma)$$

This estimate breaks down if Γ is not a simple closed curve.

Alternative blocking strategies

Construct first a partial barrier, to slow down fire propagation. Then build a wall enclosing the fire.



(unlikely to succeed, with speed $\sigma \leq 2$)