Controlling Mechanical Systems by Active Constraints

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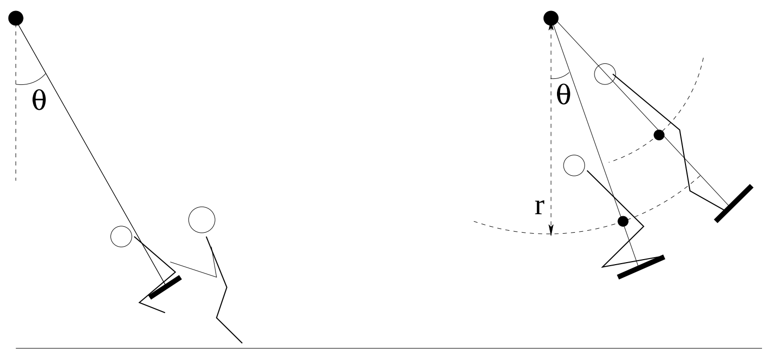
Control of Mechanical Systems: Two approaches

- by applying external forces

- by directly assigning some of the coordinates, as functions of time

Riding on a Swing



1. An external force pushing:

$$\ddot{\theta} \;=\; -\sin\theta + u(t)$$

$t \mapsto u(t)$ is the force, used to control the motion of the swing

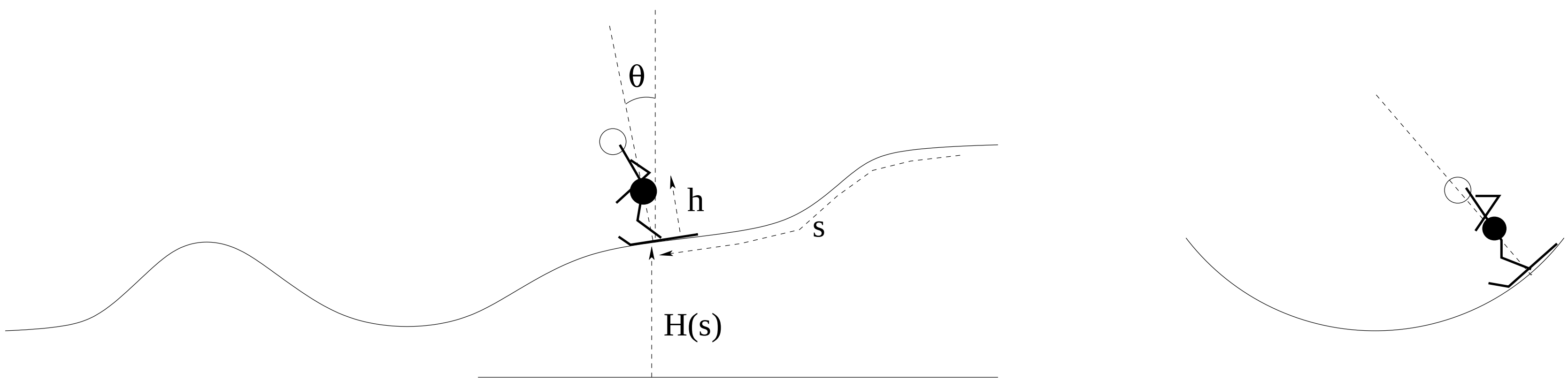
2. Changing the position of the barycenter:

r = radius of oscillation $\qquad\qquad \theta$ = angle

Assign the radius directly as function of time $r = u(t)$

$$\ddot{\theta} \;=\; -\frac{\sin\theta}{u} - \frac{2\dot{\theta}}{u}\,\dot{u}\,.$$

Skier on a narrow trail



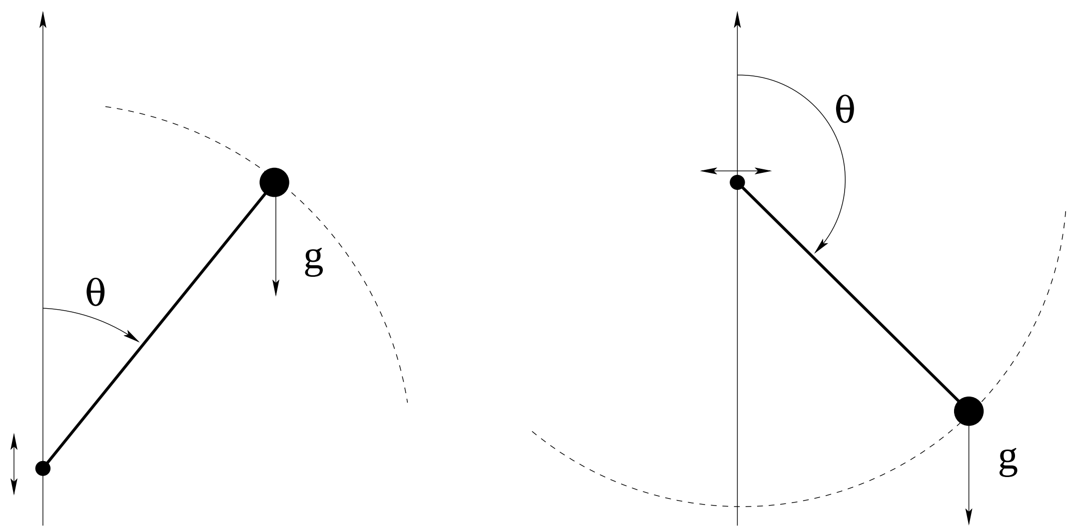
s = arc length parameter along trail

h= height of barycenter, along perpendicular line

Assign the height $h = u(t)$ as a function of time

$\Longrightarrow$ the motion $t \mapsto s(t)$ along the trail is uniquely determined

Pendulum with fixed length and oscillating pivot



For $0 < \theta < \frac{\pi}{2}$ the pendulum can be stabilized by **vertical** oscillations of the pivot

For $\frac{\pi}{2} < \theta < \pi$ the pendulum can be stabilized by **horizontal** oscillations

Swim-like motion in a perfect fluid

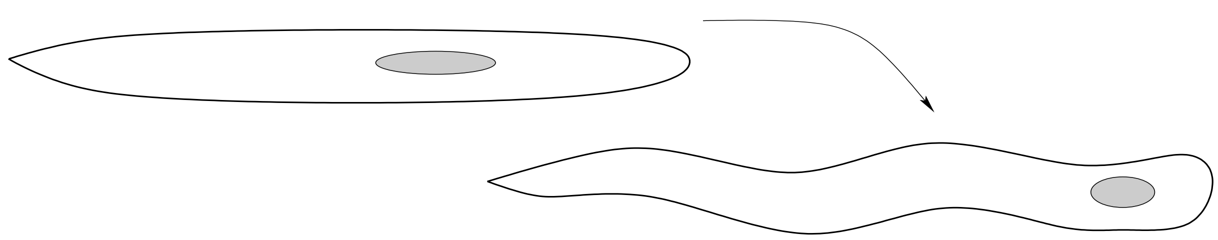
Consider:

• a deformable body whose *shape* and *internal mass distribution* are described by finitely many parameters

• immersed in a perfect fluid: *incompressible, inviscid, irrotational*

Assign some of these parameters as functions of time

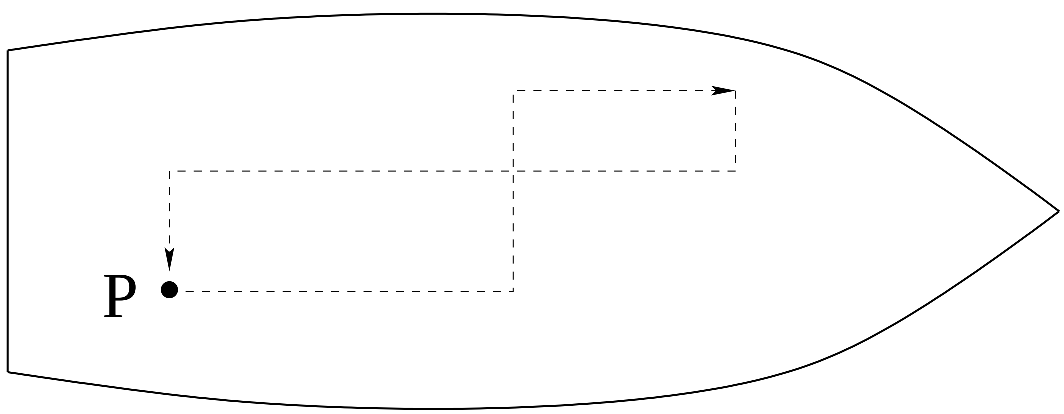
$\Longrightarrow$ determine the motion



Example (Kozlov & al., 2000 - 2003)

A point mass moving inside a rigid shell, immersed in a perfect fluid.

Assign the relative position of the point mass: $P = u(t) \in \mathbb{R}^2$



Controlling a Lagrangian system by applying external forces

Lagrangian variables: $q^1, \ldots, q^N$

$$\text{Kinetic energy:} \quad \mathcal{T}(q, \dot{q}) \; = \; \frac{1}{2} \sum_{i,j=1}^{N} g_{ij}(q) \dot{q}^i \dot{q}^j$$

Equations of motion:

$$\frac{d}{dt} \frac{\partial \mathcal{T}}{\partial \dot{q}^i} - \frac{\partial \mathcal{T}}{\partial q^i} \; = \; \phi_i(q, u(t)) \qquad\qquad i = 1, \ldots, N$$

$t \mapsto u(t) =$ control function

$\phi_i(q, u) =$ components of the external forces

Controlling a Lagrangian system by assigning some of the coordinates as functions of time

Split the coordinates in two groups:

$$q^1, \ldots, q^n, \qquad\qquad q^{n+1}, \ldots, q^{n+m}$$

Assign the last m coordinates directly as functions of time

$$q^{n+\alpha} = u_\alpha(t) \qquad\qquad \alpha = 1, \ldots, m \tag{C}$$

Find the evolution of the first n coordinates $q^1, \ldots, q^n$

Splitting of coordinates determines a **foliation:** $\mathcal{F} = \{\Lambda_u\,; \quad u \in \mathbb{R}^m\}$

Each **leaf** is a submanifold: $\Lambda_u \;=\; \Big\{(q^1, \ldots, q^n, q^{n+1}, \ldots, q^{n+m})\,; \quad q^{n+\alpha} = u_\alpha\Big\}$

At each time t, the assignment

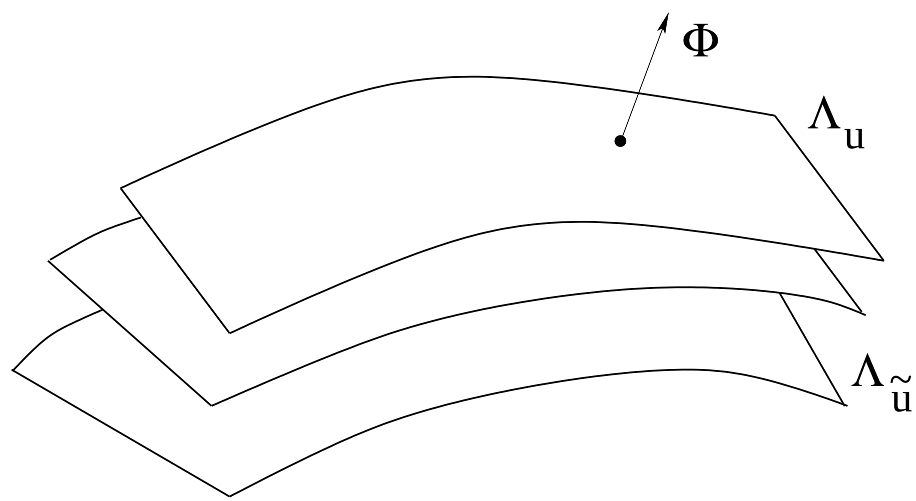
$$q^{n+\alpha} = u_\alpha(t) \qquad\qquad \alpha = 1, \ldots, m \qquad (C)$$

determines on which leaf the system is located

BASIC ASSUMPTION: the identities (C) are implemented by means of

FRICTIONLESS CONSTRAINTS

the force Φ used to implement the constraints is always
perpendicular to the leaves Λ_u (w.r.t. the metric given by the kinetic energy)



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Geometric structure:

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F. Cardin and M. Favretti, Hyper-impulsive motion on manifolds. *Dynam. Contin. Discr. Impuls. Syst.* **4** (1998), 1-21.

Analytical study of the impulsive O.D.E's:

A. Bressan and F. Rampazzo, On differential systems with vector-valued impulsive controls, *Boll. Un. Matem. Italiana* **2-B**, (1988), 641-656.

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Controllability properties

J. Baillieul, The Geometry of Controlled Mechanical Systems, in *Mathematical Control Theory*, J.Baillieul & J.C. Willems, Eds., Springer-Verlag, New York, 1998, 322-354.

A. Bressan and F. Rampazzo, Stabilization of Lagrangian systems by moving coordinates, *Arch. Rational Mech. Anal.* **196** (2010), 97-141.

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Equations of motion (without additional forces)

Hamiltonian: $$H(q,p) = \frac{1}{2}\sum_{i,j=1}^{n+m} g^{ij}(q)p_i p_j$$

conjugate momenta: $$p_i = \frac{\partial T}{\partial \dot{q}^i} = \sum_{j=1}^{n+m} g_{ij}(q)\dot{q}^j, \qquad (g^{ij}) = (g_{ij})^{-1}$$

$$\begin{cases} \dot{q}^i = \frac{\partial H}{\partial p_i} \\ \dot{p}_i = -\frac{\partial H}{\partial q^i} \end{cases} \qquad i = 1, \ldots, n$$

$$\begin{cases} \dot{q}^{n+\alpha} = \frac{\partial H}{\partial p_{n+\alpha}} \\ \dot{p}_{n+\alpha} = -\frac{\partial H}{\partial q^{n+\alpha}} + \Phi_\alpha(t) \end{cases} \qquad \alpha = 1, \ldots, m$$

For $\alpha = 1, \ldots, m$, the components of the forces $\Phi_\alpha(t)$ produced by the constraints must be determined so that $q^{n+\alpha}(t) = u_\alpha(t)$

$$\text{variables:} \qquad \begin{array}{ccc} q^1 & \dots & q^n \\ p_1 & \dots & p_n \end{array} \qquad \begin{array}{ccc} q^{n+1} & \dots & q^{n+m} \\ p_{n+1} & \dots & p_{n+m} \end{array}$$

$$\left\{ \begin{array}{rcl} \dot{q}^i & = & \frac{\partial H}{\partial p_i}(q,p) \\ \dot{p}_i & = & -\frac{\partial H}{\partial q^i}(q,p) \end{array} \right. \qquad i = 1, \dots, n$$

Solve for $q^1, \dots, q^n,\ p_1, \dots, p_n$, inserting the values

$$\left\{ \begin{array}{l} q^{n+\alpha} = u_\alpha(t) \qquad \dot{q}^{n+\alpha} = \dot{u}_\alpha(t) \\ \\ p_{n+\alpha} = p_{n+\alpha}(p_1, \dots, p_n,\ \dot{q}^{n+1}, \dots, \dot{q}^{n+m}) \end{array} \right. \qquad \alpha = 1, \dots, m$$

Analytic form of the equations

Kinetic energy matrix: $G = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} (g_{ij}) & (g_{i,n+\beta}) \\ (g_{n+\alpha,j}) & (g_{n+\alpha,\,n+\beta}) \end{pmatrix}$

$$A = \left(a^{ij}\right) \doteq (G_{11})^{-1}\,, \qquad K = \left(k^i_\alpha\right) \doteq -AG_{12}\,, \qquad B = \left(b_{\alpha,\beta}\right) \doteq G_{22} - G_{21}AG_{12}$$

Equations of motion for the free variables $q = (q^1, \ldots, q^n)$, $\quad p = (p_1, \ldots, p_n)$

$$\begin{pmatrix} \dot q \\ \\ \dot p \end{pmatrix} = \begin{pmatrix} Ap \\ \\ -\frac{1}{2}p^\dagger \frac{\partial A}{\partial q} p + F \end{pmatrix} + \begin{pmatrix} K \\ \\ -p\frac{\partial K}{\partial q} \end{pmatrix} \dot u + \dot u^\dagger \begin{pmatrix} 0 \\ \\ \frac{1}{2}\frac{\partial B}{\partial q} \end{pmatrix} \dot u\,,$$

$u = (u_1, \ldots, u_m) =$ control function $\qquad F =$ additional forces

No need to explicitly compute the forces Φ_α produced by the constraints !

Classification

$$\begin{pmatrix} \dot q \\ \\ \dot p \end{pmatrix} = \begin{pmatrix} Ap \\ \\ -\frac{1}{2}p^\dagger \frac{\partial A}{\partial q} p \end{pmatrix} + \begin{pmatrix} K \\ \\ -p\frac{\partial K}{\partial q} \end{pmatrix} \dot u + \dot u^\dagger \begin{pmatrix} 0 \\ \\ \frac{1}{2}\frac{\partial B}{\partial q} \end{pmatrix} \dot u\,,$$

1. General form: quadratic w.r.t. $\dot u$

Possible input functions:

$$u(\cdot) \in W^{1,2} = \{\text{absolutely continuous functions with } \dot u \in \mathbf{L}^2\}$$

2. Fit for jumps: affine w.r.t. $\dot u$, if $\partial B/\partial q \equiv 0$

Possible input functions:

$$u(\cdot) \in BV = \{\text{functions with bounded variation}\}$$

(assigning the path taken by the control across each jump)

3. Strongly fit for jumps: $\dot x = f(x,u)$ (with a suitable choice of coordinates)

Relevant Problems

1. Understand the relations between

- analytic form of the equation
- geometric properties of the foliation
- topologies that render continuous the "control-to-trajectory" map $u(\cdot) \mapsto q(\cdot)$

2. Representation of the dynamics in terms of a differential inclusion. Approximation with smooth controls.

3. Stabilization at a point $\bar{q}$, or around a periodic orbit.

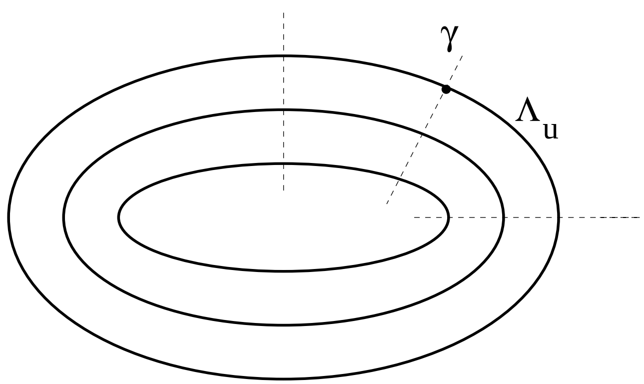
4. Optimization problems

Equivalent properties

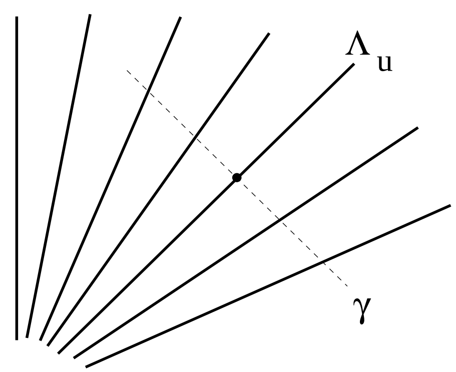
• The hyper-impulsive system is **fit for jumps**, namely $\partial B/\partial q \equiv 0$ and the equations are affine w.r.t. $\dot{u}$.

• The foliation $\{\Lambda_u\,; \quad u \in I\!R^m\}$ is **bundle like**, i.e. leaves remain at constant distance from each other (B. Reinhart, *Ann. Math.* 1959).

• Any geodesic γ that starts perpendicularly to one of the leaves, remains perpendicular to every other leaf it meets.



fit for jumps



not fit for jumps

Controlling a general system (not fit for jumps)

Interplay between linear and quadratic terms:

$$\begin{pmatrix} \dot q \\ \\ \dot p \end{pmatrix} = \begin{pmatrix} Ap \\ \\ -\frac{1}{2}p^\dagger \frac{\partial A}{\partial q} p + F \end{pmatrix} + \begin{pmatrix} K \\ \\ -p\frac{\partial K}{\partial q} \end{pmatrix} \dot u + \dot u^\dagger \begin{pmatrix} 0 \\ \\ \frac{1}{2}\frac{\partial B}{\partial q} \end{pmatrix} \dot u \,, \tag{1}$$

Reduced dynamics (neglecting linear terms):

$$\begin{pmatrix} \dot q \\ \\ \dot p \end{pmatrix} \in \begin{pmatrix} Ap \\ \\ -\frac{1}{2}p^\dagger \frac{\partial A}{\partial q} p + F \end{pmatrix} + \mathcal{V} \,, \qquad \mathcal{V} \doteq \overline{co} \left\{ w^\dagger \begin{pmatrix} 0 \\ \\ \frac{\partial B}{\partial q} \end{pmatrix} w \,; \quad w \in \mathbb{R}^m \right\} \tag{2}$$

$\mathcal{V}$ = **cone of impulses generated by control vibrations**

Theorem *(A.B. - F. Rampazzo).* Every trajectory of (2) can be uniformly approximated by trajectories of (1)

A first order reduction (slow dynamics: $p \approx 0$)

$$\begin{pmatrix} \dot q \\ \\ \dot p \end{pmatrix} = \begin{pmatrix} Ap \\ \\ -\frac{1}{2}p^\dagger \frac{\partial A}{\partial q} p \end{pmatrix} + \begin{pmatrix} K \\ \\ -p^\dagger \frac{\partial K}{\partial q} \end{pmatrix} \dot u + \dot u^\dagger \begin{pmatrix} 0 \\ \\ \frac{1}{2}\frac{\partial B}{\partial q} \end{pmatrix} \dot u \tag{1}$$

$$\dot q \in K(q,u)\dot u + \Gamma(q,u)\,, \qquad\qquad (q,u)(0) = (\bar q, \bar u) \tag{2}$$

$$\Gamma(q,u) \doteq \overline{co}\left\{ A(q,u)\left(w^\dagger \frac{\partial B}{\partial q}(q,u)\, w\right);\quad w \in I\!R^m \right\} = A\,\mathcal{V}$$

Theorem *(A.B. - Z.Wang, 2008)*. Let $t \mapsto (q^*(t), u^*(t)) \in I\!R^{n+m}$ be a trajectory of the differential inclusion (2), defined for $t \in [0,1]$.

For every $\varepsilon > 0$, there exists a smooth control $u(\cdot)$ defined on some interval $[0,T]$ such that then the corresponding solution of (1) with initial data

$$(q,u)(0) = (q^*(0), u^*(0)), \qquad p(0) = 0$$

satisfies

$$\sup_{t\in[0,T]} |p(t)| < \varepsilon, \qquad \sup_{t\in[0,T]} |q(t) - q^*(\psi(t))| < \varepsilon, \qquad \sup_{t\in[0,T]} |u(t) - u^*(\psi(t))| < \varepsilon,$$

for some increasing diffeomorphism $\psi : [0,T] \mapsto [0,1]$

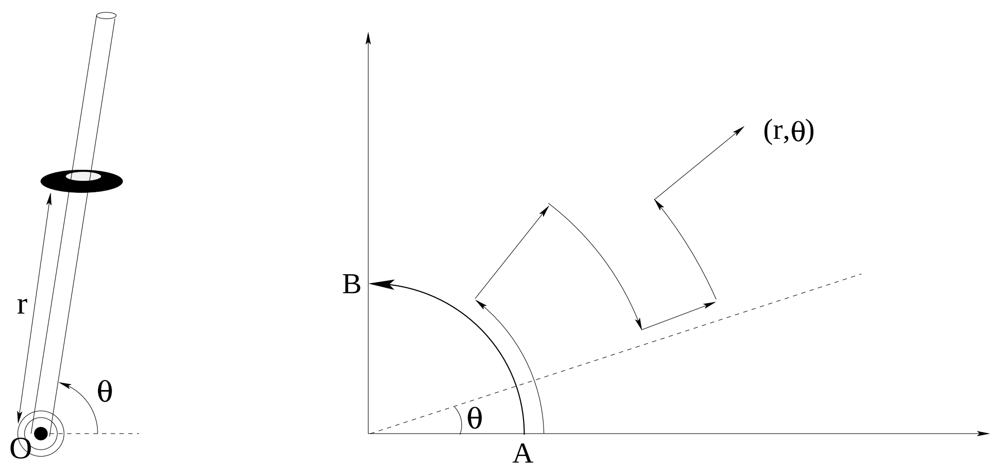
Example: bead sliding without friction along a rotating bar

$$q(t) = r = \text{radius} \qquad\qquad u(t) = \theta = \text{controlled angle}$$

$$T(r,\theta,\dot r,\dot\theta) = \frac{m}{2}(\dot r^2 + r^2\dot\theta^2) \qquad\qquad p = \frac{\partial T}{\partial \dot r} = m\dot r \qquad\qquad \begin{cases} \dot r &= p/m\,, \\ \dot p &= mr\dot u^2\,. \end{cases}$$

Every solution $t \mapsto (r^*(t), \theta^*(t))$ of the differential inclusion $\dot r^*(t) \geq 0$

can be traced by a solution of the original system, starting at rest.

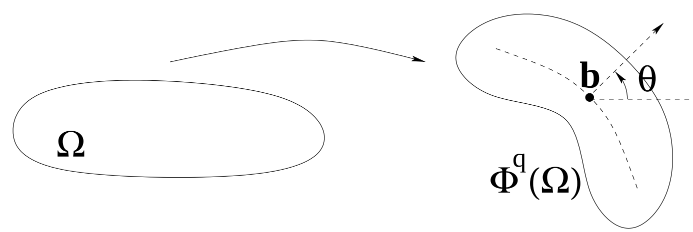


Locomotion in a Perfect Fluid

$q = (q^1, \ldots, q^N)$ = Lagrangian parameters, describing the position, mass distribution and shape of the body

$\xi \mapsto \Phi^q(\xi)$ is a volume preserving diffeomorphism

Ω = reference configuration. $\Phi^{q(t)}(\Omega)$ = region occupied by the body at time t.



For $n + m = N$, we assign the last m coordinates as functions of time, by means of **frictionless constraints**

$$q^{n+\alpha} = u_\alpha(t) \qquad\qquad \alpha = 1, \ldots, m$$

PROBLEM: Assuming that no other forces are present, describe the motion of the first n (uncontrolled) coordinates $q^1, \ldots, q^n$ and of the surrounding fluid.

$$\text{Kinetic energy of the body:} \quad \mathcal{T}^{\text{body}}(q,\dot{q}) = \sum_{i,j=1}^{N} G_{ij}(q)\dot{q}^i\dot{q}^j$$

$$\text{Kinetic energy of the surrounding fluid:} \quad \mathcal{T}^{\text{fluid}} = \int_{\mathbb{R}^d \setminus \Phi^q(\Omega)} \frac{|v(x)|^2}{2}\,dx$$

$v = v(x)$ the velocity of the fluid at the point x

Key fact: for an incompressible, non-viscous, irrotational fluid,
the velocity v is entirely determined by the finitely many parameters $q, \dot{q}$.

$$\text{Kinetic energy of the fluid:} \quad \mathcal{T}^{\text{fluid}}(q,\dot{q}) = \sum_{i,j=1}^{N} \widetilde{G}_{ij}(q)\dot{q}^i\dot{q}^j$$

The previous theory applies, with $\mathcal{T} = \mathcal{T}^{\text{body}} + \mathcal{T}^{\text{fluid}}$

Euler equations + incompressibility condition

$$v_t + v \cdot \nabla v = -\nabla p$$

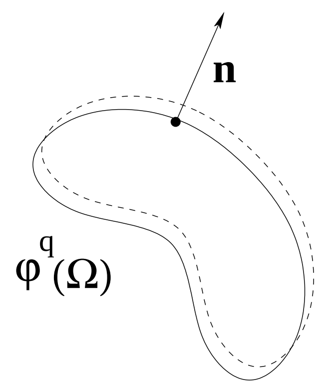
$$\operatorname{div} v \equiv 0$$

Assume: motion in $\mathbb{R}^2$, zero vorticity, zero circulation

Claim: velocity of the surrounding fluid is entirely determined by $q, \dot{q}$

Given $q, \dot{q}$, we seek the unique irrotational velocity field $v = v^{(q,\dot{q})}$ defined on $\mathbb{R}^2 \setminus \Phi^q(\Omega)$ such that

$$\left\langle v(\Phi^q(\xi)) \ , \quad \mathbf{n}^q(\xi) \right\rangle \ = \ \left\langle \sum_{i=1}^{N} \frac{\partial}{\partial q^i} \Phi^q(\xi) \cdot \dot{q}^i \ , \quad \mathbf{n}^q(\xi) \right\rangle$$



For $i \in \{1, \ldots, N\}$, $\gamma_i = \nabla \psi_i$, solve the Neumann problem in the exterior domain

$$\Delta \psi_i = 0 \qquad\qquad x \in I\!R^2 \setminus \Phi^q(\Omega)$$

$$\mathbf{n} \cdot \nabla \psi_i = \mathbf{n} \cdot \frac{\partial \Phi^q}{\partial q^i} \qquad\qquad x \in \partial \Phi^q(\Omega)$$

with $\psi_i(x) \to 0$ as $|x| \to \infty$

$$v(x) \;=\; \sum_{i=1}^{N} \gamma_i^{(q)}(x) \cdot \dot{q}^i$$

$$\int_{I\!R^2 \setminus \Phi^q(\Omega)} \frac{|v(x)|^2}{2}\, dx \;=\; \sum_{i,j=1}^{N} \widetilde{G}_{ij}(q) \dot{q}^i \dot{q}^j$$

- Computation of the coefficients $\widetilde{G}_{ij}(q)$ may be hard !

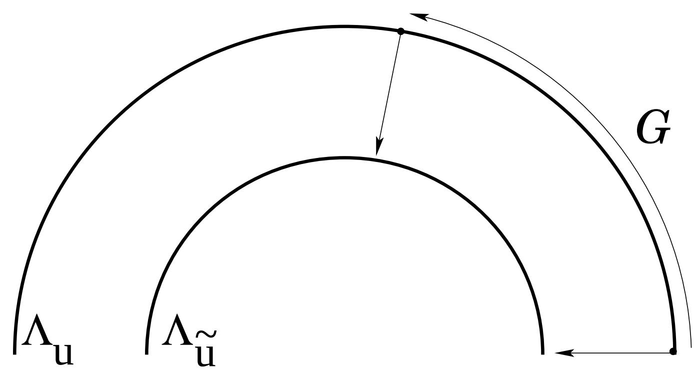
Approximate expansion: C.Grotta Ragazzo, *SIAM J. Appl. Math.*, 2003.

Symmetries

distance between leaves is constant $\quad\Longleftrightarrow\quad$ fit for jumps

Theorem *(A.Arsie, 2008).* Assume: there exists a symmetry group $\mathcal{G}$ whose orbits are the leaves Λ_u, and which preserves the metric g_{ij}.

Then the system is fit for jumps



Geometry of swim-like motion

Take $\mathcal{G}$ = group of translations and rigid rotations

Variable splitting: $q = (q^1, \ldots, q^n, q^{n+1}, \ldots, q^m)$

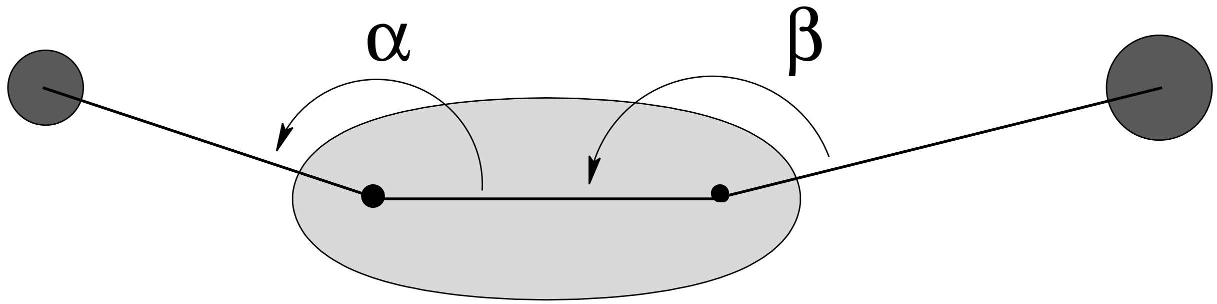
• If the controlled variables $q^{n+1}, \ldots, q^{n+m}$ entirely determine the shape of the body and the distribution of masses, up to a translation and a rigid rotation, then the system [body + surrounding fluid] is **fit for jumps**

• In general, if the controlled variables $q^{n+1}, \ldots, q^{n+m}$ do not entirely determine the shape of the body, then the system [body + surrounding fluid] is **not fit for jumps**

- in the presence of freely flapping fins

- two or more swimmers

Snake-like chain immersed in a perfect fluid



Angles α, β are assigned as functions of time

- System is fit for jumps
- Completely controllable

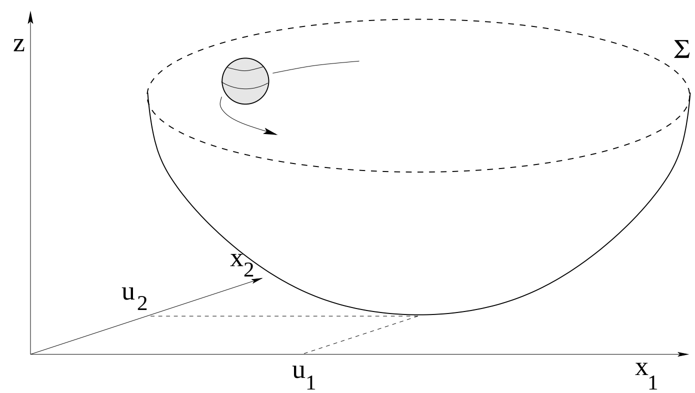
Additional non-holonomic constraints

(A.B. - F.Rampazzo - K.Han, 2010)

$$q = (q^1, \dots, q^n, q^{n+1}, \dots, q^{n+m})$$

active constraints: $\qquad q^{n+\alpha} = u_\alpha(t), \qquad \alpha = 1, \dots, m$

additional non-holonomic constraints: $\displaystyle\sum_{i=1}^{n+m} \omega_{ki}(q)\,\dot{q}^i = 0 \qquad k = 1, \dots, \nu$



- equations of motion
- geometric structure $\iff$ fit for jumps

$\mathcal{M}$ a Riemann manifold with metric corresponding to the kinetic energy

$$\mathcal{T}(q, \dot{q}) \;=\; \sum_{ij} g_{ij}(q)\, \dot{q}^i \dot{q}^j$$

Γ = non-integrable distribution describing the non-holonomic constraint

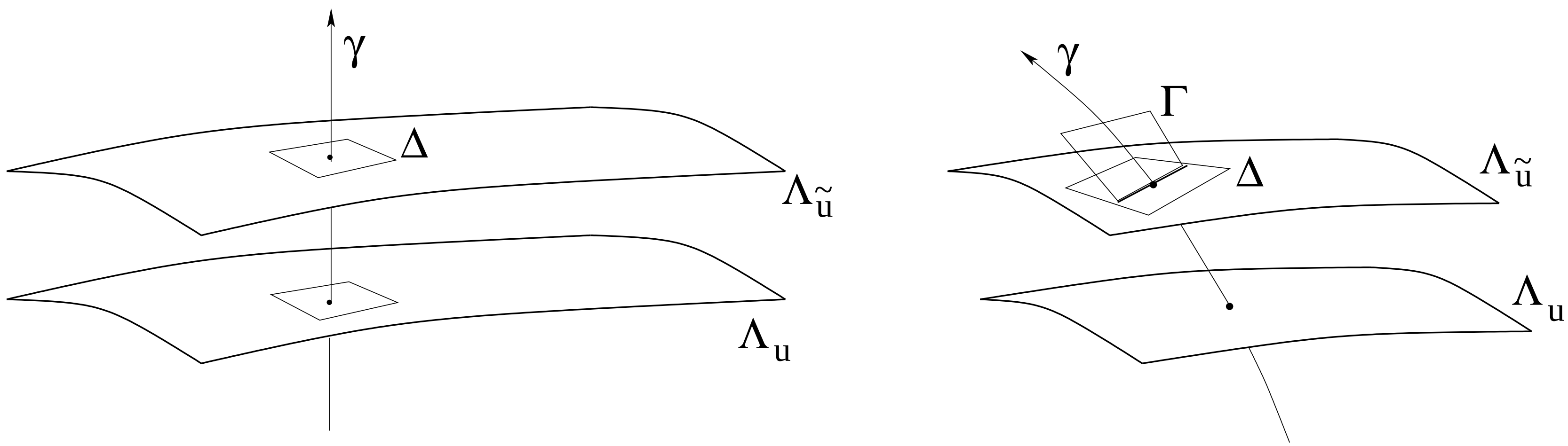
Definition. A Γ-geodesic is a solution to

$$\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{q}} - \frac{\partial \mathcal{T}}{\partial q} \;\in\; \Gamma^{ker} \qquad\qquad \dot{q} \in \Gamma$$

i.e., a trajectory of the system with constraints, without external forces

Δ = integrable distribution tangent to the foliation

Γ = non-integrable distribution describing the non-holonomic constraint



Theorem *(A.B. - F.Rampazzo, 2010)*

The system is fit for jumps **if and only if**

the orthogonal bundle $(\Delta \cap \Gamma)^{\perp}$ is Γ-geodesically invariant