Traffic models on a network of roads

Alberto Bressan

Department of Mathematics, Penn State University

bressan@math.psu.edu

A brief review of some recent work

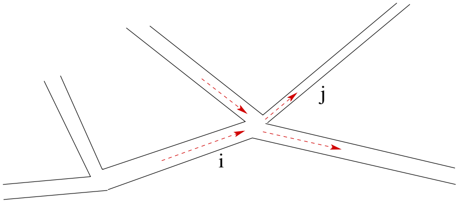
$$\rho_t + [\rho\, v(\rho)]_x \;=\; 0 \qquad (LWR)$$

conservation law on each road $+$ junction conditions

- examples of ill-posedness for some models based on Riemann Solvers
- well-posedness in $\mathbf{L}^\infty$ for junction models including buffers
- existence results for global optima and Nash equilibria on a network of roads

Modeling traffic flow at a junction

incoming roads: $i \in \mathcal{I}$ outgoing roads: $j \in \mathcal{O}$



Boundary conditions account for:

- θ_{ij} = fraction of drivers from road i that turn into road j
- c_i = relative priority of drivers from road i
  (fraction of time drivers from road i get green light, on average)

$$\sum_j \theta_{ij} = 1$$

- H.Holden, N.H.Risebro, A mathematical model of traffic flow on a network of unidirectional roads, *SIAM J. Math. Anal.* **26**, 1995.
- G.M.Coclite, M.Garavello, B.Piccoli, Traffic flow on a road network, *SIAM J. Math. Anal.* **36**, 2005.
- M.Herty, S.Moutari, M.Rascle, Optimization criteria for modeling intersections of vehicular traffic flow, *Netw. Heterog. Media* **1**, 2006.
- M.Garavello, B.Piccoli, Conservation laws on complex networks, *Ann.I.H.Poincaré* **26** 2009.
- M.Garavello, B.Piccoli, *Traffic Flow on Networks*, AIMS, 2006.
- A.B., S.Canic, M.Garavello, M.Herty, and B.Piccoli, Flow on networks: recent results and perspectives, *EMS Surv. Math. Sci.* **1** (2014), 47–111.

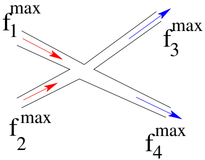
Construction of a Riemann Solver (Coclite, Garavello, Piccoli ...)

$$\begin{cases} \rho_1, \ldots, \rho_N = \text{initial densities} \quad \text{(constant on each road)} \\ \theta_{ij} = \text{fraction of drivers from road } i \text{ that turn into road } j \end{cases}$$

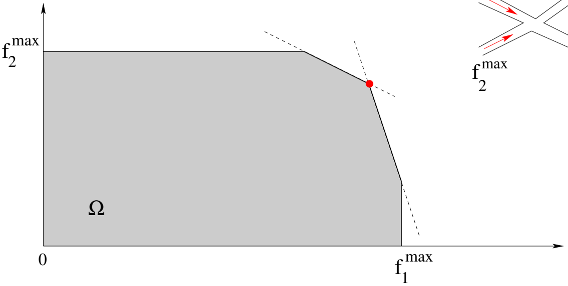
$$\begin{cases} f_i^{max} = \text{maximum flux on the incoming road } i \in \mathcal{I} \\ f_j^{max} = \text{maximum flux on the outgoing road } j \in \mathcal{O} \end{cases}$$

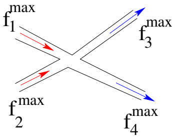
Feasible region $\Omega \subset \mathbb{R}^n$. Vector of incoming fluxes $(f_1, \ldots, f_n) \in \Omega$ iff

- $f_i \in [0, f_i^{max}] \qquad i \in \mathcal{I}$
- $\sum_i f_i \theta_{ij} \leq f_j^{max} \qquad j \in \mathcal{O}$



The feasible region Ω





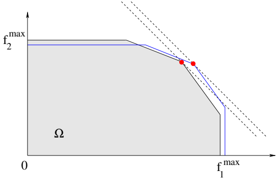
Riemann solver $\iff$ rule for selecting a point in the feasible region Ω

Example: maximize the total flux through the node: $\sum_{i \in \mathcal{I}} f_i$

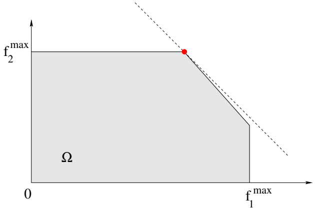
Continuity of the Riemann Solver

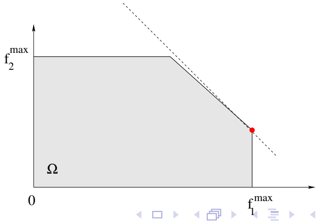
Selection rule: maximize the total flux $\sum_{i \in \mathcal{I}} f_i$

- If the turning preferences θ_{ij} remain constant, the fluxes f_i depend Lipschitz continuously on the Riemann data $\rho_1, \ldots, \rho_N$.



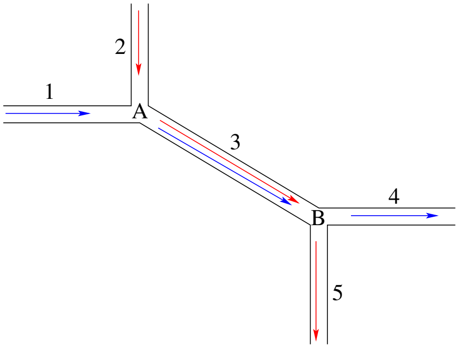
- The Riemann solver is discontinuous w.r.t. changes in the θ_{ij}





Why should the θ_{ij} vary in time?

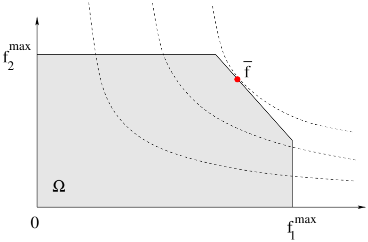
Drivers' turning preferences θ_{ij} must be determined as part of the solution



# of vehicles on road i that wish to turn into road j is conserved:

$$(\rho\theta_{ij})_t + (\rho v_i(\rho)\theta_{ij})_x \;=\; 0$$

Continuous Riemann Solvers (A.B. - F.Yu, *Discr. Cont. Dyn. Syst.*, 2015)



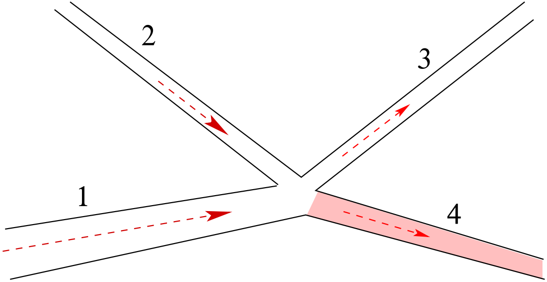
- The selection rule: maximize $\prod_{i\in\mathcal{I}} f_i$ yields a Riemann solver which is **Hölder continuous** w.r.t. all variables

$$(\rho_i, \theta_{ij})_{i\in\mathcal{I}, j\in\mathcal{O}} \mapsto (f_i)_{i\in\mathcal{I}}$$

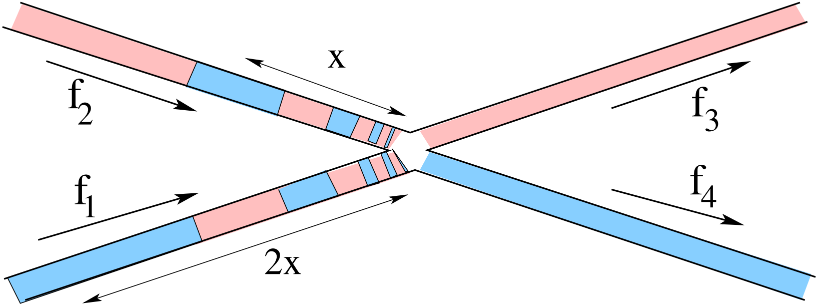
- One can also construct a Riemann solver which is **Lipschitz continuous** w.r.t. all variables
- Unfortunately all this is useless, because if the θ_{ij} are allowed to vary **the Cauchy problem is ill posed** anyway

Modeling assumptions

- If all cars arriving at the intersection can immediately move to outgoing roads, no queue is formed.
- If outgoing roads are congested, the inflow of cars from road 1 is twice as large as the inflow from road 2.



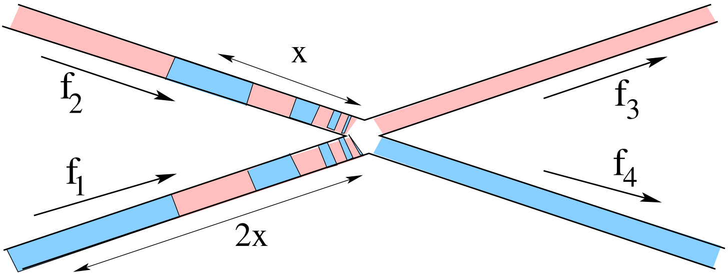
Example 1: θ_{ij} with unbounded variation, two solutions



$$f_k(\rho) \;=\; 2\rho - \rho^2 \qquad \text{maximum flux on every road:} \quad f_k^{max} = 1$$

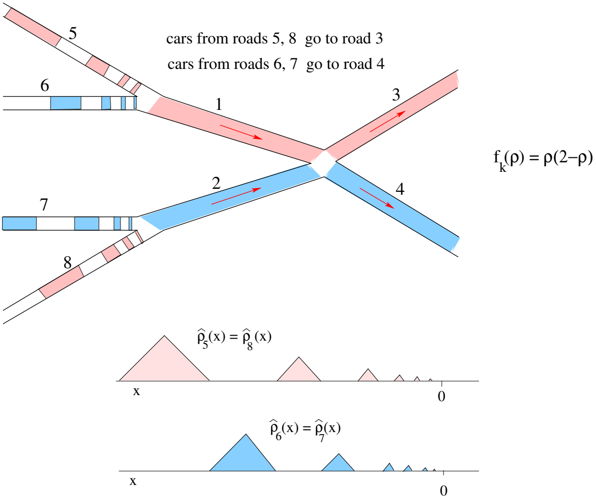
Initial data: $\rho_k \;=\; 1, \qquad k = 1, 2, 3, 4$

$$\hat{\theta}_{13}(x) \;=\; \hat{\theta}_{24}(x) \;=\; \begin{cases} 1 & \text{if} \quad -2^{-n} < x < -2^{-n-1}, \qquad n \text{ even} \\ 0 & \text{if} \quad -2^{-n} < x < -2^{-n-1}, \qquad n \text{ odd} \end{cases}$$



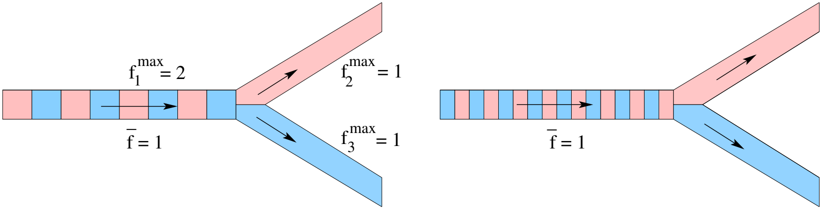
Solution 1. Incoming fluxes: $f_1(t,0) = 1, \quad f_2(t,0) = 1$

Solution 2. Incoming fluxes: $f_1(t,0) = \frac{2}{3}, \quad f_2(t,0) = \frac{1}{3}$

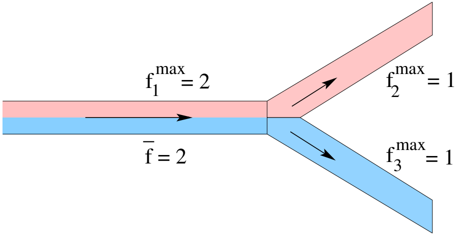


At some time $T > 0$, the same initial data as in Example 1 is created at the junction of roads 1 and 2

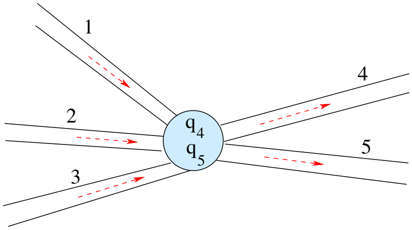
Example 3: lack of continuity w.r.t. weak convergence



As $n \to \infty$, the weak limit is $\theta_{12} = \theta_{13} = \frac{1}{2}$



An intersection model with buffers



- the intersection contains a buffer with finite capacity (a traffic circle)
- $t \mapsto q_j(t) =$ queues in front of outgoing roads $j \in \mathcal{O}$, within the buffer
- incoming drivers are admitted to the intersection at a rate depending on the size of these queues
- drivers already inside the intersection flow out to the road of their choice at the fastest possible rate

- M. Herty, J. P. Lebacque, and S. Moutari, A novel model for intersections of vehicular traffic flow. *Netw. Heterog. Media* 2009.
- M. Garavello and P. Goatin, The Cauchy problem at a node with buffer. *Discrete Contin. Dyn. Syst.* 2012.
- M. Garavello and B. Piccoli, A multibuffer model for LWR road networks, in *Advances in Dynamic Network Modeling in Complex Transportation Systems, 2013.*

Toward the analysis of global optima and Nash equilibria, we need

- well posedness for $\mathbf{L}^\infty$ initial data ρ_k^0, θ_{ij}^0
- continuity of travel time w.r.t. weak convergence

$$\left\{\begin{array}{rcll} \rho_{k,t} + f_k(\rho_k)_x & = & 0 & \qquad \text{conservation laws} \\ \theta_{ij,t} + v_i(\rho_i)\theta_{ij,x} & = & 0 & \qquad \text{linear transport equations} \end{array}\right.$$

Intersection models with buffers

(A.B., K.Nguyen, *Netw. Heter. Media*, 2015)

$q_j(t)$ = size of the queue, inside the intersection, of cars waiting to enter road j

(SBJ) - Single Buffer Junction

$M > 0$ = maximum number of cars that can occupy the intersection

$c_i > 0$, $i \in \mathcal{I}$, priorities given to different incoming roads

Incoming fluxes $\bar{f}_i$ satisfy $$\bar{f}_i \;\leq\; c_i\Big(M - \sum_{j\in\mathcal{O}} q_j\Big), \qquad\qquad i \in \mathcal{I}$$

Well-posedness of the Cauchy problem with buffers

Theorem (A.B.- K.Nguyen, *Netw. Heter. Media* **10** (2015), 255-293).

Consider an intersection modeled by (SBJ).

For any $\mathbf{L}^\infty$ initial data $\rho(0,x) = \bar\rho_k(x) \in [0, \rho_k^{jam}]$,

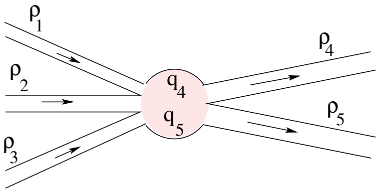
$$q_j(0) = \bar q_j\,, \qquad \theta_{ij}(0,x) = \bar\theta_{ij} \in [0,1] \qquad \text{with } \sum_{j\in\mathcal{O}} \bar q_j < M, \qquad \sum_{j\in\mathcal{O}} \bar\theta_{ij}(x) = 1$$

the Cauchy problem has a unique entropy admissible solution, defined for all $t \geq 0$.

Moreover, the travel times depend continuously on the initial data, in the topology of weak convergence.

$$\bar\rho_k^n(x) \rightharpoonup \bar\rho_k \qquad \bar\rho_i^n \bar\theta_{ij}^n \rightharpoonup \bar\rho_i \bar\theta_{ij}, \qquad \bar q_j^n \to \bar q_j$$

Variational formulation



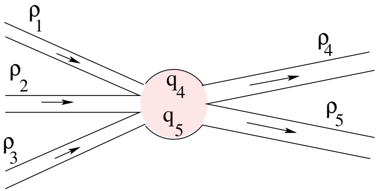
$$(q_4, q_5) \xrightarrow{\text{Lax formula}} (V_1, V_2, V_3, V_4, V_5) \xrightarrow{\text{boundary values}} (q_4, q_5)$$

length of queues

$$V_k(t,x) = \int^x \rho_k(t,x)\, dx$$

- If the queue sizes $q_j(t)$ within the buffer are known, then the initial-boundary value problems can be independently solved along each incoming road.

  These solutions can be computed by solving suitable variational problems.
  From the value functions V_k, the traffic density $\rho_k = V_{k,x}$ along each incoming or outgoing road is recovered by a Lax type formula.



$$(q_4, q_5) \xrightarrow{\text{Lax formula}} (V_1, V_2, V_3, V_4, V_5) \xrightarrow{\text{boundary values}} (q_4, q_5)$$

length of queues

$$\rho_k = V_{k,x}$$

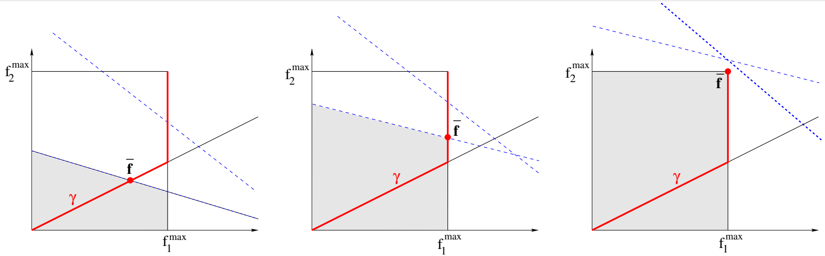
traffic densities

- Conversely, if these value functions V_k are known, then the queue sizes q_j can be determined by balancing the boundary fluxes of all incoming and outgoing roads
- The solution of the Cauchy problem is obtained as the unique fixed point of a contractive transformation
- The present model accounts for backward propagation of queues along roads leading to a crowded intersection, it achieves well-posedness for general $\mathbf{L}^\infty$ data, and continuity of travel time w.r.t. weak convergence

The limit Riemann Solver for buffer of vanishing size

Theorem (A.B., A.Nordli, *Netw. Heter. Media*, to appear).

Letting the size of the buffer $M \to 0$ one obtains a Riemann Solver which is **Lipschitz continuous** w.r.t. all variables ρ_i, θ_{ij}



$$s \mapsto \gamma(s) = (f_1(s), \ldots, f_m(s)), \qquad f_i(s) \doteq \min\{c_i s\,,\; f_i^{max}\}$$

Then the incoming fluxes are $\bar{f}_i = f_i(\bar{s})$

$$\text{where:} \qquad \bar{s} = \max\left\{ s \geq 0\,;\; \sum_{i\in\mathcal{I}} f_i(s)\,\theta_{ij} \leq f_j^{max} \quad \text{for all } j \in \mathcal{O} \right\}$$

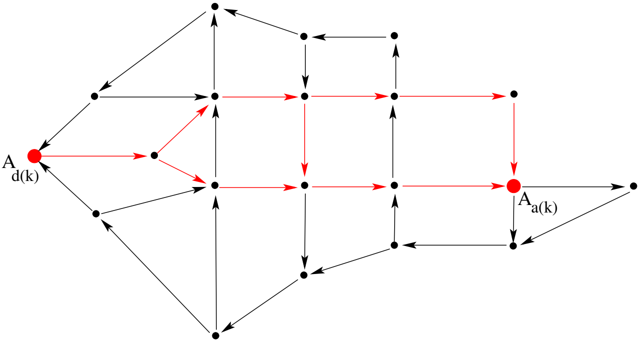
Optimization Problems for Traffic Flow on a Network

n groups of drivers with different origins and destinations, and different costs

Drivers in the k-th group depart from $A_{d(k)}$ and arrive to $A_{a(k)}$

can use different paths $\Gamma_1, \Gamma_2, \ldots$ to reach destination

Departure cost: $\varphi_k(t)$ arrival cost: $\psi_k(t)$



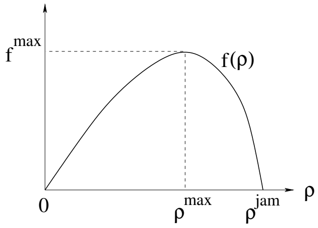
Basic assumptions

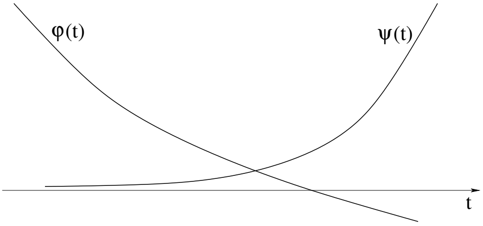
(A1) The flux functions $\rho \mapsto f_i(\rho) = \rho\, v(\rho)$ are all strictly concave down.

$$f_i(0) \;=\; f_i(\rho_i^{jam}) \;=\; 0\,, \qquad\qquad f_i'' \;<\; 0\,.$$

(A2) For each group of drivers $k = 1, \ldots, N$, the cost functions φ_k, ψ_k satisfy

$$\varphi_k' \;<\; 0, \qquad \psi_k, \psi_k' \;<\; 0, \qquad\qquad \lim_{|t|\to\infty} \Big(\varphi_k(t) + \psi_k(t)\Big) \;=\; +\infty$$





Optima and Equilibria

An admissible family $\{\bar{u}_{k,p}\}$ *of departure rates is* **globally optimal** *if it minimizes the sum of the total costs of all drivers*

$$J(\bar{u}) \doteq \sum_{k,p} \int \Big(\varphi_k(t) + \psi_k(\tau_p(t))\Big) \bar{u}_{k,p}(t)\, dt$$

An admissible family $\{\bar{u}_{k,p}\}$ *of departure rates is a* **Nash equilibrium** *if no driver of any group can lower his own total cost by changing departure time or switching to a different path to reach destination.*

$$\varphi_k(t) + \psi_k(\tau_p(t)) = C_k \qquad \text{for all } t \in Supp(\bar{u}_{k,p})$$

$$\varphi_k(t) + \psi_k(\tau_p(t)) \geq C_k \qquad \text{for all } t \in \mathbb{R}$$

Existence results

Theorem (A.B.- K.Nguyen, *Netw. Heter. Media* **10** (2015), 717–748).

Under the assumptions (A1)-(A2), on a general network of roads, there exists at least one globally optimal solution.

If, in addition, the travel time admits a uniform upper bound, then a Nash equilibrium exists.

- Proof is achieved by finite dimensional approximations
  + a topological argument (relying on the continuity of the travel time w.r.t. weak convergence of the departure rates)
- For a single group of drivers on a single road, solutions are unique.
  Uniqueness is not expected to hold, on a general network.
- An earlier existence result was proved in
  A.B. - Ke Han, Netw. Heter. Media, 2013,
  with simplified boundary conditions at road intersections.
- Necessary conditions for global optimality?